Dong Hoon Lee
Department of Mathematics
Case Western Reserve University

The structure of complex Lie groups

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To my teacher

Karl Heinrich Hofmann

Preface

These notes aim at providing an introductory study of the structure theory of complex Lie groups. A great portion of this material derives from a course on the subject which was given at TU Darmstadt, Germany, during the summer term of 1997.

Complex Lie groups have been used often as auxiliaries in the study of real Lie groups in such areas as differential geometry and representation theory. These notes take up the subject as a primary objective for which the structural aspects of complex Lie groups are emphasized and explored systematically. Our exposition follows, in essence, the earlier work of Hochschild and Mostow, who used the theory of representative functions of Lie groups as an invaluable tool in the study of Lie groups. Their studies rest, for example, on the notion of the universal algebraic hull of a Lie group, which is built on the algebra of the representative functions and which provides an important linkage between Lie groups and algebraic groups. This enables us to combine the algebraic group technique with that of the Lie theory so as to provide an efficient tool for this exposition.

The following is a brief description of each chapter. The first chapter covers the general, introductory concepts of complex Lie groups. Chapter 2 contains the theory of representative functions of Lie groups, which arises from the study of representations of Lie groups, and we obtain, in Chapter 3, some key results related to the extension problem of representations, which become essential tools for the subsequent chapters. Chapter 4 is devoted entirely to the study of the structure of complex Lie groups, and Chapter 5 deals with the question of when a complex subgroup of a complex Lie group carries the structure of an affine algebraic group that is compatible with its analytic structure. In Chapter 6 we study the observable subgroups of complex Lie groups, and we describe them in terms of their maximal algebraic subgroups. In addition, two appendices have been added at the end of the book for reference. We assemble, in Appendix A, some basic concepts and results from the theory of Lie algebras, while Appendix B carries a brief account of theory of pro-affine algebraic groups. Throughout the notes the reader's familiarity with the basic theory of *real* Lie groups as well as a knowledge of some elementary results from the theory of affine algebraic groups are presupposed.

It is a pleasure to thank my colleagues at TU Darmstadt for their invitation, and I am especially indebted to Karl Hofmann for his help and encouragement in preparing these notes. I am most grateful to Ta-Sun Wu for many useful discussions and also for years of collaboration. I would also like to acknowledge support from Case Western Reserve University, the Deutsche Forschungsgemeinschaft, and the Global Analysis Research Center of Seoul National University during several stages in the preparation of these notes.

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Dong Hoon Lee

Contents

Chapter 1

Complex Lie Groups

In this chapter the basic notion of complex Lie groups is introduced, and some of the essential tools that will be used in the remaining chapters are developed ([2], [8], [22]).

1.1 Analytic Manifolds

Complex Analytic Manifolds We first recall the definition of an analytic function of n variables. Let $\mathrm{U} \subset \mathbb{C}^n$ be an open set. A function $f : \mathrm{U} \to \mathbb{C}$ is called *complex analytic* (or *holomorphic*) on U if, given any $(a_1, \cdots, a_n) \in U$, there exist a positive number η and a power series

$$\sum_{s_1,\cdots,s_n \geq 0} c_{s_1,\cdots,s_n}(x_1 - a_1)^{s_1} \cdots (x_n - a_n)^{s_n}, \ c_{s_1,\cdots,s_n} \in \mathbb{C}$$

around $(a_1, \cdots, a_n)$ such that the series converges absolutely and uniformly to the sum $f(x_1, \cdots, x_n)$ for all n-tuples $(x_1, \cdots, x_n)$ with $|x_i - a_i| < \eta, \ 1 \leq i \leq n$.

Let M be a Hausdorff topological space. By an *open chart* on M, we mean a pair (U, φ), where U is an open subset of M and φ is a homeomorphism of U onto an open subset of $\mathbb{C}^n$. A *complex analytic structure* on M of dimension n is a collection of open charts (U_i, φ_i), $i \in I$, such that the following conditions are satisfied:

(i) $M = \bigcup_{i \in I} U_i$;

(ii) For each pair $i, j \in I$, the map

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is complex analytic;

(iii) The collection (U_i, φ_i), $i \in I$, is complete in the sense that it is a maximal family of open charts for which (i) and (ii) hold.

The collection (U_i, φ_i), $i \in I$, satisfying the conditions (i) and (ii) is called an *atlas* on M, and its members (U_i, φ_i) are called *local charts.* As usual the condition (iii) is not essential in the definition because an atlas can be extended uniquely to a larger atlas satisfying all three conditions. Given a local chart (U_i, φ_i) at p, the system of functions

$$u_1 \circ \varphi_i, \cdots, u_n \circ \varphi_i,$$

where $u_1, \cdots, u_n$ denote the coordinate functions of $\mathbb{C}^n$, is called a *local coordinate system* in U_i.

A *complex (analytic) manifold of dimension* n is a connected Hausdorff space M together with a complex analytic structure of dimension n. A $\mathbb{C}$-valued function f defined on an open set W in a complex manifold M is called *complex analytic* on W if, for each $p \in W$, there is a local chart (U_i, φ_i) at p with $U_i \subset W$ such that $f \circ \varphi_i^{-1}$ is complex analytic on $\varphi_i(U_i)$.

Given a complex manifold M, $p \in M$ and two functions f and g, each defined and complex analytic on an open set containing p, we write $f \sim g$ if f and g coincide in an open neighborhood of p. The relation $\sim$ defines an equivalence relation among all functions, which are defined and complex analytic locally at p, and its equivalence classes are called the *germs of complex analytic functions at* p . In the usual way, they form a $\mathbb{C}$-algebra, which we denote by $\mathcal{F}_M(p)$, or simply $\mathcal{F}(p)$. For any function f locally complex analytic at a point p, we denote the germ of f (i.e., the equivalence class containing f) by f_p. A *tangent vector* to M at p is a differentiation of $\mathcal{F}(p)$ into $\mathbb{C}$, i.e., a $\mathbb{C}$-linear map $\tau : \mathcal{F}(p) \to \mathbb{C}$ such that

$$\tau(f_p g_p) = \tau(f_p) g(p) + f(p) \tau(g_p)$$

for $f_p, g_p \in \mathcal{F}(p)$. The tangent vectors to M at p form a complex linear space, called the *tangent space* to M at p, and we denote it by $T_p(M)$.

From here on the germ f_p of a function f at p will be written simply f, unless there is confusion in doing so. Thus, for example, the differentiation condition above will be stated as

$$\tau(fg) = \tau(f) g(p) + f(p) \tau(g)$$

for $f,\ g \in \mathcal{F}(p)$.

Let M and M' be complex analytic manifolds. A map ϕ from M into M' is said to be *complex analytic* at $p \in M$ if, for every local chart (U_i, φ_i) of M at p and every local chart (V_j, ψ_j) of M' at $\phi(p)$ such that $\phi(U_i) \subset V_j$, the map

$$\psi_j \circ \phi \circ \varphi_i^{-1} : \varphi_i(U_i) \to \psi_j(V_j)$$

is complex analytic, or equivalently, if the map $f \circ \phi$ is complex analytic at p whenever f is a complex analytic function at $\phi(p)$. The map ϕ is called *complex analytic* if it is so at each point of M.

Suppose $\phi : M \to N$ is complex analytic at $p \in M$. For each $\tau \in T_p(M)$, $f \mapsto \tau(f \circ \phi) : \mathcal{F}(\phi(p)) \to \mathbb{C}$ defines a tangent vector τ' of N at $\phi(p)$, and

$$\tau \mapsto \tau' : T_p(M) \to T_{\phi(p)}(N)$$

defines a $\mathbb{C}$-linear map. This map is called the *differential* of ϕ at p, and we denote it by $d\phi_p$, or simply $d\phi$ whenever p is understood.

Product Manifolds Let M and N be a product of two complex analytic manifolds of dimension m and n, respectively, and suppose that M and N are equipped with atlases $\mathcal{U} = \{(U_i, \varphi_i) : i \in I\}$ and $\mathcal{V} = \{(V_j, \psi_j) : j \in J\}$, respectively. For each pair (i, j), the map

$$\varphi_i \times \psi_j : U_i \times V_j \to \mathbb{C}^m \times \mathbb{C}^n = \mathbb{C}^{m+n}$$

is a homeomorphism of the open set $U_i \times V_j$ of $M \times N$ onto an open set of $\mathbb{C}^{m+n}$, and the product space $M \times N$ may be given the unique complex analytic structure of dimension $m+n$ for which all the pairs

$$(U_i \times V_j,\ \varphi_i \times \psi_j),\ (i, j) \in I \times J$$

are local charts. We call this the *product* of the complex manifolds M and N. Thus a $\mathbb{C}$-valued function f, defined on an open set W of $M \times N$, is complex analytic if, for each $(p, q) \in W$, there are local charts (U, φ), (V, ψ) of M, N at p, q, respectively, such that $U \times V \subset W$ and that

$$f \circ (\varphi \times \psi)^{-1} : \varphi(U) \times \psi(V) \to U \times V \to \mathbb{C}$$

is complex analytic.

For any fixed point $(p, q) \in M \times N$, define the maps

$$\alpha : M \to M \times N; \quad \beta : N \to M \times N$$

by $\alpha(x) = (x, q),\ x \in M$ and $\beta(y) = (p, y),\ y \in N$. Then the map

$$(\sigma, \tau) \mapsto d\alpha_p(\sigma) + d\beta_q(\tau) : T_p(M) \oplus T_q(N) \to T_{(p,q)}(M \times N)$$

is a $\mathbb{C}$-linear isomorphism. We shall often identify both spaces by means of this isomorphism.

Below we shall briefly review the tangent vectors and the tangent space of differentiable manifolds.

Let M be a differentiable or a real analytic manifold, and let $p \in M$. For the subsequent use, we briefly describe a basis for the tangent space $T_p(M)$ of M at p using a local coordinate system. Let $\mathcal{F}(p)$ denote the $\mathbb{R}$-algebra of (the germs of) real analytic functions at p. $T_p(M)$ is, by definition, the $\mathbb{R}$-linear space consisting of all differentiations of $\mathcal{F}(p)$ into $\mathbb{R}$. Let $x_1, \cdots, x_n$ be a local coordinate system on a local chart (U_i, φ_i) at p. For $1 \leq j \leq n$, let

$$(\frac{\partial}{\partial x_j})_p : \mathcal{F}(p) \to \mathbb{R}$$

denote the map

$$f \mapsto (\frac{\partial (f \circ \varphi_i^{-1})}{\partial t_j})_{\varphi_i(p)}.$$

Then

$$(\frac{\partial}{\partial x_1})_p, \cdots, (\frac{\partial}{\partial x_n})_p$$

form a basis for $T_p(M)$.

Now we return to a complex analytic manifold M of dimension n with a complete atlas $(U_i, \varphi_i),\ i \in I$. For $p \in M$, choose a local chart (U_i, φ_i) at p, and let $z_1, \cdots, z_n$ be a local coordinate system on U_i so that

$$\varphi_i(q) = (z_1(q), \cdots, z_n(q))$$

for $q \in U_j$. We write each complex-valued function z_j in the form $x_j + \sqrt{-1}y_j$ where x_j and y_j are real-valued functions. The map $\varphi_i' : U_i \to \mathbb{R}^{2n}$ given by

$$\varphi_i'(q) = (x_1(q), \ldots, x_n(q), y_1(q), \ldots, y_n(q))$$

is a homeomorphism of U_i onto an open subset $\mathbb{R}^{2n}$. The collection of open charts $\{U_i, \varphi_i'), i \in I\}$ on M turns M into a *real* analytic manifold, which is denoted by $M_{\mathbb{R}}$. This real analytic structure on M is said to be *underlying* the complex analytic structure above. The $2n$ vectors

$$(\frac{\partial}{\partial x_1})_p, \cdots, (\frac{\partial}{\partial x_n})_p;\ (\frac{\partial}{\partial y_1})_p, \cdots, (\frac{\partial}{\partial y_n})_p$$

form a basis for $T_p(M_{\mathbb{R}})$.

Tangent Vectors of Complex Manifolds Let M be a complex analytic manifold. We examine the relationship between *complex* tangent vectors in $T_p(M)$ and *real* tangent vectors in $T_p(M_{\mathbb{R}})$. Let $\mathcal{F}(p)$ (resp. $\mathcal{F}_{\mathbb{R}}(p)$) denote the $\mathbb{C}$-algebra (resp. $\mathbb{R}$-algebra) of the germs of all complex analytic functions (resp. real analytic functions) at p. Then the complexification $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{F}_{\mathbb{R}}(p)$ of $\mathcal{F}_{\mathbb{R}}(p)$ is a $\mathbb{C}$-algebra that contains $\mathcal{F}(p)$ as a subalgebra. We would like to extend each $\tau \in T_p(M)$ to a differentiation

$$\tau' : \mathbb{C} \otimes_{\mathbb{R}} \mathcal{F}_{\mathbb{R}}(p) \to \mathbb{C}.$$

In order to achieve this, we choose an open neighborhood U of p and a local coordinate system $z_1, \cdots, z_n$ on U. Write each z_j in the form $x_j + \sqrt{-1}y_j$ as before.

For $\tau \in T_p(M)$, we define

$$\tau' : \mathcal{F}_{\mathbb{R}}(p) \to \mathbb{R}$$

as follows. First set

$$\begin{aligned} \tau'(x_j) &= \frac{1}{2}(\tau(z_k) + \overline{\tau(z_k)}) \\ \tau'(y_j) &= \frac{i}{2}(\overline{\tau(z_k)} - \tau(z_k)), \end{aligned}$$

and then define

$$\tau' = \sum_j \tau'(x_j)(\frac{\partial}{\partial x_j})_p + \sum_j \tau'(y_j)(\frac{\partial}{\partial y_j})_p. \tag{1.1.1}$$

Noting that $\tau'(x_j),\ \tau'(y_j) \in \mathbb{R}$, we see that τ' is a differentiation, i.e., $\tau' \in T_p(M_{\mathbb{R}})$, and τ' in turn has a canonical extension to a differentiation (denoted also by τ') of the $\mathbb{C}$-algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{F}_{\mathbb{R}}(p)$:

$$\tau' = I \otimes \tau' : \mathbb{C} \otimes_{\mathbb{R}} \mathcal{F}_{\mathbb{R}}(p) \to \mathbb{C}.$$

If we put

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - \sqrt{-1}\frac{\partial}{\partial y_j})$$
$$\frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial y_j}),$$

then we may express τ' as

$$\tau' = \sum_j \tau(z_j)(\frac{\partial}{\partial z_j})_p + \sum_j \overline{\tau(z_j)}(\frac{\partial}{\partial \overline{z}_j})_p.$$

Note that the conditions $\frac{\partial f}{\partial \overline{z}_j} = 0$ are exactly the Cauchy-Riemann equations. Thus the complex tangent vectors $(\frac{\partial}{\partial z_j})_p$ form a basis for $T_p(M)$, and on the subalgebra $\mathcal{F}(p)$ of $\mathbb{C} \otimes_{\mathbb{R}} \mathcal{F}_{\mathbb{R}}(p)$, we have

$$\tau (= \tau') = \sum_j \tau(z_j)(\frac{\partial}{\partial z_j})_p. \qquad (1.1.2)$$

We also deduce the following proposition from the equation (1.1.1).

Proposition 1.1 *Let M be a complex analytic manifold, and let $p \in M$. The map $\tau \mapsto \tau'$ defines an $\mathbb{R}$-linear isomorphism between $T_p(M)$ and $T_p(M_{\mathbb{R}})$.* ∎

This proposition enables us to adopt the following convention. Whenever we view a complex analytic manifold M as a real analytic manifold (i.e., $M = M_{\mathbb{R}}$), then we regard $T_p(M)$ as the tangent space of the real analytic manifold M.

Complex Analytic Vector Fields Let M be a complex analytic manifold. A *vector field* X on M is a rule which assigns to each point p of M a tangent vector $X_p \in T_p(M)$. Let U be an open subset of M, and let $f : U \to \mathbb{C}$ be a complex analytic function. For $p \in U$, we put $Xf(p) = X_p(f_p)$. Then $p \mapsto Xf(p)$ defines a function $Xf : U \to \mathbb{C}$. A vector field X is called *complex analytic* if, for each complex analytic function f (on some open subset U), the function Xf is also complex analytic on U.

If U is an open subset of M on which there is a local coordinate system $z_1, \cdots, z_n$, then, for each $1 \le j \le n$, we obtain a complex analytic vector field $\frac{\partial}{\partial z_j}$, given by

$$p \mapsto (\frac{\partial}{\partial z_j})_p$$

on U. Thus any vector field X on U may be written in the form

$$\sum_{j=1}^{n} f_j \frac{\partial}{\partial z_j}, \qquad (1.1.3)$$

where f_j, $1 \le j \le n$, are functions defined on U. The functions f_j are called *the components* of X with respect to the coordinate system $z_1, \cdots, z_n$. If X is complex analytic, then the functions f_j are all complex analytic on U because $f_j = X z_j$. Conversely, if $f_1, \cdots, f_n$ are complex analytic on U, then clearly $X = \sum_j f_j \frac{\partial}{\partial z_j}$ is a complex analytic vector field on U.

For two vector fields X and Y on M, we define a vector field $[X, Y]$ on M by

$$[X, Y] = X \circ Y - Y \circ X,$$

that is,

$$[X, Y]_p(f) = (X(Yf) - Y(Xf))(p), \ p \in M, \ f \in \mathcal{F}(p).$$

That $[X, Y]$ is indeed a vector field (i.e., $[X, Y]_p \in T_p(M)$ for all p) is straightforward to verify. If X and Y are complex analytic, so is $[X, Y]$. In fact, let $p \in M$, and let U be an open neighborhood of p with a local coordinate system $z_1, \cdots, z_n$ at p. If h is a complex analytic function on U, then, using (1.1.3), we may express Xh and Yh on U by

$$Xh = \sum_j f_j \frac{\partial h}{\partial z_j}; \ Yh = \sum_j g_j \frac{\partial h}{\partial z_j},$$

where the f_j and g_j are all defined and complex analytic on U. Then an easy computation shows that

$$[X, Y]h = \sum_j k_j \frac{\partial h}{\partial z_j}$$

where k_j is given by

$$k_j = \sum_i (g_i \frac{\partial f_j}{\partial z_i} - f_i \frac{\partial g_j}{\partial z_i}).$$

Since the functions $f_i,\ g_i$ are all complex analytic on U, so are the functions k_j, proving that $[X, Y]h$ is complex analytic on U.

Let $\Gamma(M)$ be the set of all complex analytic vector fields on M. Clearly $\Gamma(M)$ is a complex linear space under the usual addition and scalar multiplication. Moreover, for $X,\ Y \in \Gamma(M)$, we have shown above that $[X, Y] \in \Gamma(M)$, and thus we see that $\Gamma(M)$ is a complex Lie algebra (of infinite dimension).

Almost Complex Structure An *almost complex structure* on a differentiable manifold M (of *class* C^∞) is a rule which assigns to each point $p \in M$ an $\mathbb{R}$-linear endomorphism J_p of the tangent space $T_p(M)$ such that $J_p^2 = -I$ for each $p \in M$. A differentiable manifold with a fixed almost complex structure is called an *almost complex manifold.*

As a simple example of an almost complex manifold, we consider $M = \mathbb{R}^2$, viewed as a differentiable manifold with coordinate system (x, y). For each $p \in M$, the $\mathbb{R}$-linear map

$$J_p : T_p(M) \to T_p(M),$$

given by

$$a(\frac{\partial}{\partial x})_p + b(\frac{\partial}{\partial y})_p \mapsto -b(\frac{\partial}{\partial x})_p + a(\frac{\partial}{\partial y})_p,$$

where $a, b \in \mathbb{R}$, defines an almost complex structure on M.

Below we shall show that every complex manifold (when viewed as a real manifold) admits a canonical almost complex structure. To that end, we first consider the space $\mathbb{C}^n$ with the complex coordinates $(z_1, \cdots, z_n)$ and write $z_j = x_j + \sqrt{-1}y_j$, where $x_j,\ y_j \in \mathbb{R}$. With respect to the real coordinate system $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ of the $2n$-dimensional differentiable manifold $\mathbb{C}^n$, we get the *canonical almost complex structure* J on $\mathbb{C}^n$, which is determined by

$$J_p((\frac{\partial}{\partial x_j})_p) = (\frac{\partial}{\partial y_j})_p;\ J_p((\frac{\partial}{\partial y_j})_p) = -(\frac{\partial}{\partial x_j})_p,\ 1 \le j \le n. \quad (1.1.4)$$

Lemma 1.2 *A differentiable map ϕ of an open subset U of $\mathbb{C}^n$ into $\mathbb{C}^m$ is complex analytic if and only if its differential preserves the canonical almost complex structures on $\mathbb{C}^n$ and $\mathbb{C}^m$, i.e., $d\phi_p \circ J_p = J_{\phi(p)} \circ d\phi_p$ for all $p \in U$.*

Proof. Let $(z_1, \cdots, z_n)$ and $(w_1, \cdots, w_m)$ be the natural coordinate systems in $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, and put $z_k = x_k + \sqrt{-1}y_k$, $(1 \le k \le n)$, and $w_k = u_k + \sqrt{-1}v_k$, $(1 \le k \le m)$. We express $\phi(z_1, \cdots, z_n) = (w_1, \cdots, w_m)$ in terms of these real coordinates:

$$\begin{aligned} u_k &= u_k(x_1, \cdots, x_n, y_1, \cdots, y_n); \\ v_k &= v_k(x_1, \cdots, x_n, y_1, \cdots, y_n), \end{aligned}$$

$1 \le k \le m$, where the functions u_k and v_k are differentiable. ϕ is complex analytic if and only if all w_k are complex analytic functions of $z_1, \cdots, z_n$, and each w_k is a complex analytic function of $z_1, \cdots, z_n$ if and only if it is a complex analytic function of each variable z_k by a well-known classical theorem of several complex variables. It follows from the Cauchy-Riemann equations that ϕ is complex analytic if and only if we have

$$\begin{aligned} \frac{\partial u_k}{\partial x_j} - \frac{\partial v_k}{\partial y_j} &= 0 \\ \frac{\partial u_k}{\partial y_j} + \frac{\partial v_k}{\partial x_j} &= 0 \end{aligned}$$

for $1 \le j \le n$, $1 \le k \le m$.

On the other hand, the chain rule provides

$$\begin{aligned} d\phi(\frac{\partial}{\partial x_j}) &= \sum_{k=1}^{m}(\frac{\partial u_k}{\partial x_j})(\frac{\partial}{\partial u_k}) + \sum_{k=1}^{m}(\frac{\partial v_k}{\partial x_j})(\frac{\partial}{\partial v_k}) \qquad (1.1.5) \\ d\phi(\frac{\partial}{\partial y_j}) &= \sum_{k=1}^{m}(\frac{\partial u_k}{\partial y_j})(\frac{\partial}{\partial u_k}) + \sum_{k=1}^{m}(\frac{\partial v_k}{\partial y_j})(\frac{\partial}{\partial v_k}) \end{aligned}$$

for $1 \le j \le k$. It follows from the formulas (1.1.4) and (1.1.5) that $d\phi \circ J = J \circ d\phi$ if and only if ϕ is complex analytic. ■

Almost Complex Structure on Complex Manifolds Let M be a complex analytic manifold. We now define an almost complex structure on M by transferring the almost complex structure on $\mathbb{C}^n$ to M by means of charts. Indeed, choose a local chart (U_i, φ_i) at $p \in M$, and define the $\mathbb{R}$-linear endomorphism

$$J_p^M : T_p(M) \to T_p(M)$$

by

$$J_p^M = d\varphi_i^{-1} \circ J_{\varphi_i(p)} \circ d\varphi_i$$

where J is the canonical almost complex structure on $\mathbb{C}^n$. In terms of the local coordinate system $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ on U_i, J^M is given by

$$J_p^M((\frac{\partial}{\partial x_j})_p) = (\frac{\partial}{\partial y_j})_p; \; J_p^M((\frac{\partial}{\partial y_j})_p) = -(\frac{\partial}{\partial x_j})_p \qquad (1.1.6)$$

where $1 \le j \le n$. To show that the endomorphism J_p^M is independent of choice of local charts at p, suppose (U_j, φ_j) is another chart at p. Then the function $\varphi_j \circ \varphi_i^{-1}$ is complex analytic, and hence the differential $d(\varphi_j \circ \varphi_i^{-1}) = d\varphi_j \circ d\varphi_i^{-1}$ commutes with J (Lemma 1.2):

$$J_{\varphi_j(p)} \circ (d\varphi_j \circ d\varphi_i^{-1}) = (d\varphi_j \circ d\varphi_i^{-1}) \circ J_{\varphi_i(p)}.$$

Consequently, we have

$$d\varphi_i^{-1} \circ J_{\varphi_i(p)} \circ d\varphi_i = d\varphi_j^{-1} \circ J_{\varphi_j(p)} \circ d\varphi_j,$$

proving that J_p^M is independent of choice of charts. Clearly we have $(J_p^M)^2 = -I$, and we see that the assignment $J^M : p \mapsto J_p^M$ is an almost complex structure on M. We call J^M the *canonical almost complex structure* on the complex manifold M.

Remark 1.3 Let J be the canonical almost complex structure J on a complex analytic manifold M. The effect of J on each tangent space $T_p(M)$ is exactly the multiplication by $\sqrt{-1}$, i.e.,

$$J_p(\tau) = \sqrt{-1}\tau, \; \tau \in T_p(M). \qquad (1.1.7)$$

To see this, choose a local coordinate system $z_1, \cdots, z_n$ at p; $z_j = x_j + \sqrt{-1}y_j$, and express τ as

$$\tau = \sum_j c_j (\frac{\partial}{\partial z_j})_p,$$

where $c_j \in \mathbb{C}$. If $c_j = a_j + \sqrt{-1}b_j$ with $a_j,\ b_j \in \mathbb{R}$, the corresponding real tangent vector $\tau' \in T_p(M_{\mathbb{R}})$ (see Proposition 1.1) under the identification $T_p(M) = T_p(M_{\mathbb{R}})$ is

$$\tau' = \sum_j a_j(\frac{\partial}{\partial x_j})_p + \sum_j b_j(\frac{\partial}{\partial y_j})_p,$$

and

$$\begin{aligned} J_p(\tau) &= J_p(\sum_j a_j(\frac{\partial}{\partial x_j})_p + \sum_j b_j(\frac{\partial}{\partial y_j})_p) \\ &= \sum_j a_j J_p((\frac{\partial}{\partial x_j})_p) + \sum_j b_j J_p((\frac{\partial}{\partial y_j})_p) \\ &= \sum_j a_j(\frac{\partial}{\partial y_j})_p + \sum_j b_j(-\frac{\partial}{\partial x_j})_p \\ &= \sum_j (a_j\sqrt{-1}(\frac{\partial}{\partial z_j})_p - b_j(\frac{\partial}{\partial z_j})_p) \\ &= \sqrt{-1}\tau, \end{aligned}$$

proving the assertion. ■

We deduce the following proposition easily from Lemma 1.2.

Proposition 1.4 *Let M and M' be complex manifolds, and let J and J' be the canonical almost complex structures on M and M', respectively. A differentiable map $\phi : M \to M'$ is complex analytic if and only if $d\phi \circ J = J' \circ d\phi$, i.e., for each $p \in M$,*

$$d\phi_p \circ J_p = J'_{\phi(p)} \circ d\phi_p.$$

■

As an immediate consequence of Proposition 1.4, we have

Corollary 1.5 *Two complex analytic manifolds, which have the same underlying differentiable manifold, are identical if the corresponding almost complex structures coincide.* ■

1.2 Complex Lie Groups

Complex Lie Groups By a *complex analytic group*, we mean a group G, which is also a complex analytic manifold, such that the group multiplication $(x, y) \mapsto xy : G \times G \to G$ and the inversion $x \mapsto x^{-1} : G \to G$ are complex analytic. A *complex Lie group* is defined to be a topological group G such that the identity component G_0 is open in G and is given the structure of a complex analytic group and also that the conjugations effected by the elements of G on G_0 are complex analytic automorphisms of G_0. We here note that the second condition is automatically satisfied if G is connected.

The Lie Algebra of a Lie Group A vector field X on a complex analytic group G is called *left-invariant* if it satisfies

$$dL_a(X_x) = X_{ax}$$

for all a, $x \in G$, where $L_a : G \to G$ denotes the left translation $x \mapsto ax$ by a.

Lemma 1.6 *Any left-invariant vector field X on a complex analytic group G is complex analytic.*

Proof. In view of the left invariance, it is enough to show that X is analytic at the identity element 1. Choose a neighborhood U of 1 in G on which there is a local coordinate system $z_1, \cdots, z_n$. Thus our proof amounts to showing that, for $1 \leq j \leq n$, the components of X

$$x \mapsto X_x(z_j)$$

are complex analytic at 1. Let V be an open neighborhood of 1 such that $V^2 \subset U$. Since the map $(x, y) \mapsto z_j(xy) : V \times V \to \mathbb{C}$ is complex analytic on $V \times V$, there is a function $F(u_1, \cdot\cdot, u_n, w_1, \cdot\cdot, w_n)$, which is defined and complex analytic on some open neighborhood of $(z_1(1), \cdot\cdot, z_n(1), z_1(1), \cdot\cdot, z_n(1))$ in $\mathbb{C}^{2n}$ such that

$$z_j(xy) = F(z_1(x), \cdots, z_n(x), z_1(y), \cdots, z_n(y)).$$

We have

$$X_x(z_j) = dL_x X_1(z_j) = X_1(z_j \circ L_x) = \sum_k X_1(z_k)(\frac{\partial g(y)}{\partial z_k})_{y=1},$$

where $g(y) = z_j \circ L_x(y) = z_j(xy),\ y \in V$. We have, for each k,

$$\left(\frac{\partial g(y)}{\partial z_k}\right)_{y=1} = \left(\frac{\partial F}{\partial w_k}\right)_{(x,1)},$$

where the indices $(x, 1)$ mean that the partial derivative is taken at the point $(z_1(x), \cdots, z_n(x), z_1(1), \cdots, z_n(1))$ corresponding to $(x, 1)$. Thus we obtain

$$X_x(z_j) = \sum_k X_1(z_k)\left(\frac{\partial F}{\partial w_k}\right)_{(x,1)}.$$

Since, for $1 \leq k \leq n$, the function $x \mapsto (\frac{\partial F}{\partial w_k})_{(x,1)}$ is complex analytic at 1 and $X_1(z_k)$ is constant, it follows from the above expression that $x \mapsto X_x(z_j)$ is complex analytic. ■

The left-invariant analytic vector fields on a complex analytic group G form a Lie subalgebra of the complex Lie algebra $\Gamma(G)$. We call it the *Lie algebra* of G and denote it by $\mathcal{L}(G)$. In the case G is a complex Lie group, the Lie algebra of G refers to the Lie algebra of the identity component of G, i.e., $\mathcal{L}(G) = \mathcal{L}(G_0)$.

Theorem 1.7 *Let G be a complex analytic group. Then the map*

$$X \mapsto X_1 : \mathcal{L}(G) \to T_1(G)$$

is a $\mathbb{C}$-linear isomorphism. In particular, $\dim_{\mathbb{C}} \mathcal{L}(G) = \dim G$.

Proof. The map is clearly $\mathbb{C}$-linear. If $X \in \mathcal{L}(G)$ with $X_1 = 0$, then for any $x \in G$, $X_x = dL_x(X_1) = 0$, proving that $X = 0$. Thus the map is injective. Suppose $\tau \in T_1(G)$, and define, for any $x \in G$, the tangent vector $X_x = dL_x(\tau)$. The assignment $x \mapsto X_x$ defines a left-invariant vector field X on G, which is complex analytic by Lemma 1.6, i.e., $X \in \mathcal{L}(G)$, and $X_1 = \tau$, proving that the map is surjective. ■

Let G and H be complex analytic groups. For a complex analytic homomorphism $\phi : G \to H$, the differential $d\phi_1 : T_1(G) \to T_1(H)$ is a $\mathbb{C}$-linear map, which, by Theorem 1.7, induces a unique $\mathbb{C}$-linear map

$$d\phi : \mathcal{L}(G) \to \mathcal{L}(H)$$

so that the diagram

$$\begin{array}{ccc} \mathcal{L}(G) & \xrightarrow{d\phi} & \mathcal{L}(H) \\ \downarrow & & \downarrow \\ T_1(G) & \xrightarrow[d\phi_1]{} & T_1(H) \end{array}$$

is commutative, where the vertical maps are the isomorphisms of Theorem 1.7. We call $d\phi$ the *differential* of ϕ . It follows immediately from the left-invariance of X that $d\phi_x(X_x) = (d\phi(X))_{\phi(x)}$ for all $X \in \mathcal{L}(G)$ and $x \in G$.

Theorem 1.8 *If $\phi : G \to H$ is a complex analytic homomorphism, then its differential $d\phi : \mathcal{L}(G) \to \mathcal{L}(H)$ is a homomorphism of Lie algebras.*

Proof. For $X,\ Y \in \mathcal{L}(G)$, we need to show

$$d\phi[X, Y] = [d\phi(X), d\phi(Y)].$$

In light of Theorem 1.7, it is enough to show

$$(d\phi[X, Y])_1 = [d\phi(X), d\phi(Y)]_1.$$

Let g be a function defined and complex analytic in an open neighborhood V of 1 in H. Then $g \circ \phi$ is also analytic on $U = \phi^{-1}(V)$, and, for any $Z \in \mathcal{L}(G)$, we have

$$Z(g \circ \phi) = (d\phi(Z))(g) \circ \phi. \tag{1.2.1}$$

In fact, for all $y \in U$,

$$\begin{aligned} Z(g \circ \phi)(y) &= Z_y(g \circ \phi) \\ &= d\phi_y(Z_y)(g) \\ &= (d\phi(Z))_{\phi(y)}(g) \\ &= (d\phi(Z)(g)) \circ \phi(y), \end{aligned}$$

establishing (1.2.1). Applying (1.2.1) to X and Y repeatedly, we get

$$\begin{aligned} [X, Y](g \circ \phi) &= (X \circ Y - Y \circ X)(g \circ \phi) \\ &= d\phi(X)(d\phi(Y)(g)) \circ \phi - d\phi(Y)(d\phi(X)(g)) \circ \phi \\ &= [d\phi(X), d\phi(Y)](g) \circ \phi. \end{aligned}$$

Now

$$\begin{aligned}(d\phi[X,Y])_1(g) &= d\phi_1([X,Y]_1)(g)\\ &= [X,Y](g\circ\phi)(1)\\ &= ([d\phi(X),d\phi(Y)](g))\circ\phi(1)\\ &= [d\phi(X),d\phi(Y)]_1(g),\end{aligned}$$

proving $(d\phi[X,Y])_1 = [d\phi(X), d\phi(Y)]_1$. ■

Adjoint Representation For a complex Lie group G with Lie algebra $\mathfrak{g}$ and $x \in G$, we define $I_x : G \to G$ by $I_x(y) = xyx^{-1}$, $y \in G$. This is an automorphism of G, and its restriction to the identity component G_0 is an automorphism of the complex analytic group G_0. Let Ad(x) denote the differential $d(I_x) : \mathfrak{g} \to \mathfrak{g}$. Then $Ad(x)$ is an automorphism of the complex Lie algebra $\mathfrak{g}$, and $x \mapsto Ad(x)$ defines the *adjoint representation* $Ad_G : G \to GL(\mathfrak{g}, \mathbb{C})$ of G. If G is viewed as a real Lie group, Ad is a (real) analytic homomorphism, and its differential is the adjoint representation $ad : \mathcal{L}(G) \to \mathfrak{gl}(\mathfrak{g}, \mathbb{C})$. As we shall see later, Ad is complex analytic.

Just as in the case of real Lie groups, complex Lie groups form a category in which morphisms are complex analytic homomorphisms, and $G \rightsquigarrow \mathcal{L}(G)$ defines a functor from the category of complex Lie groups to the category of complex Lie algebras. Although the theory of real analytic groups carries over to complex analytic groups in many cases, there are, however, some important differences. For example, a closed connected subgroup of a complex analytic group is always real analytic but not necessarily complex analytic, and a continuous homomorphism from a complex analytic group to another such may not be complex analytic, even though it is real analytic. As described in what follows, we shall deal with such differences by viewing a complex analytic group as a real analytic group with additional structure.

Complex Structure of Real Analytic Groups Given a finite-dimensional $\mathbb{R}$-linear space V, an $\mathbb{R}$-linear map $J : V \to V$ is called a *complex structure* of V if $J^2 = -I_V$. A real linear space V equipped with a complex structure J can be made into a complex linear space $\widehat{V}$ by defining the scalar multiplication

$$(a + \sqrt{-1}b)v = av + J(bv) \text{ for all } a, b \in \mathbb{R},\ v \in V.$$

The space $\widehat{V}$ is called the *complex linear space associated with the complex structure* J of V. A *complex (Lie algebra) structure* of a real Lie algebra $\mathfrak{g}$ is a complex structure J on the $\mathbb{R}$-linear space $\mathfrak{g}$ satisfying

$$[u, J(v)] = J([u, v]) \quad \forall u, v \in \mathfrak{g}, \tag{1.2.2}$$

that is, $J \circ ad(u) = ad(u) \circ J$ for every $u \in \mathfrak{g}$, where ad denotes the adjoint representation of the Lie algebra $\mathfrak{g}$. In this case, the associated complex linear space $\widehat{\mathfrak{g}}$ can be made into a complex Lie algebra with the bracket operation inherited from the real Lie algebra $\mathfrak{g}$. In fact, all we need to check is the $\mathbb{C}$-bilinearity of the bracket operation $(u, v) \mapsto [u,\ v]$. Since the bracket operation is already $\mathbb{R}$-bilinear, it suffices to show

$$[\sqrt{-1}u, v] = \sqrt{-1}[u, v] = [u, \sqrt{-1}v].$$

Since $\sqrt{-1}u = J(u),\ u \in \mathfrak{g}$, the above follows from (1.2.2). The complex Lie algebra $\widehat{\mathfrak{g}}$ hence obtained is called the *complex Lie algebra associated with the complex structure* J of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a complex Lie algebra. The Lie algebra $\mathfrak{g}$, when viewed as a real Lie algebra by restricting the action of $\mathbb{C}$ to $\mathbb{R}$, becomes a real Lie algebra, which we denote by $\mathfrak{g}_{\mathbb{R}}$. The map $u \mapsto \sqrt{-1}u$ defines a morphism

$$J : \mathfrak{g}_{\mathbb{R}} \to \mathfrak{g}_{\mathbb{R}}$$

of real Lie algebras. This is a complex structure of the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ in the sense defined above, and the associated complex Lie algebra is $\mathfrak{g}$ itself, i.e., $\widehat{\mathfrak{g}_{\mathbb{R}}} = \mathfrak{g}$.

Now let G be a complex analytic group with its Lie algebra $\mathfrak{g}$. The group G together with the underlying real analytic structure of the complex analytic manifold G becomes a real analytic group, which we denote by $G_{\mathbb{R}}$. The tangent space isomorphisms (Proposition 1.1) at the various points of G are compatible with group translations of G, and we thus have

Proposition 1.9 *Let G be a complex analytic group. There is a natural isomorphism of real Lie algebras*

$$\mathcal{L}(G)_{\mathbb{R}} \cong \mathcal{L}(G_{\mathbb{R}})$$

∎

Whenever convenient, we identify $\mathcal{L}(G)_{\mathbb{R}}$ with $\mathcal{L}(G_{\mathbb{R}})$ under the isomorphism above. Thus Proposition 1.9 states that, when the complex analytic group G is viewed as a real analytic group (i.e., $G = G_{\mathbb{R}}$), its Lie algebra $\mathcal{L}(G)$ is a real Lie algebra, which is equipped with a complex structure, namely, the multiplication by $\sqrt{-1}$. As we shall see in the following two theorems, the complex structure on a complex analytic group and a morphism of complex analytic groups can be interpreted entirely in terms of almost complex structure on their underlying real analytic groups.

Theorem 1.10 *Let G be a complex analytic group with Lie algebra $\mathfrak{g}$. The canonical almost complex structure on $G_{\mathbb{R}}$ induces a complex structure J of the real Lie algebra $\mathfrak{g} = \mathcal{L}(G_{\mathbb{R}})$, and it coincides with the natural complex structure on $\mathfrak{g}$ (i.e., multiplication by $\sqrt{-1}$).*

Proof. Let J be the canonical almost complex structure on G. For $a \in G$, the left translation $L_a : x \mapsto ax$ and the inner automorphism $I_a : x \mapsto axa^{-1}$ are both complex analytic, and their differentials dL_a and $Ad(a) = dI_a$ therefore commute with J by Proposition 1.4. Hence, for any $X \in \mathcal{L}(G_{\mathbb{R}}) = \mathfrak{g}$ (i.e., left-invariant vector field X on $G_{\mathbb{R}}$), the vector field JX, defined by $x \mapsto (JX)_x = J_x(X_x)$, is also left invariant, and hence $X \mapsto JX$ defines an $\mathbb{R}$-linear endomorphism J of $\mathfrak{g}$ with $J^2 = -I$. Since $Ad(a) \circ J = J \circ Ad(a)$ for all $a \in G$, we have $J \circ ad(X) = ad(X) \circ J$ for all $X \in \mathfrak{g}$. Consequently, $J : X \mapsto JX$ is a complex structure on the real Lie algebra $\mathcal{L}(G_{\mathbb{R}}) = \mathfrak{g}$.

For the second assertion, let $X \in \mathfrak{g}$ and let $a \in G$. By Remark 1.3, we have

$$(JX)_a = J_a(X_a) = \sqrt{-1}X_a = (\sqrt{-1}X)_a,$$

proving $JX = \sqrt{-1}X$. ∎

The complex structure on $\mathfrak{g}$ in Theorem 1.10 is called the *complex structure induced* by the almost complex structure J. If $\phi : G \to H$ is a morphism of complex analytic groups, then, viewed as a morphism of real analytic groups, its differential commutes with the canonical almost complex structures of G and H by Proposition 1.4. From this we easily deduce

Theorem 1.11 *Let G and H be complex analytic groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$,respectively, and let $\phi : G \to H$ be a morphism of* *complex analytic groups. If J_G (resp. J_H) is the complex structure on $\mathfrak{g}$ (resp. $\mathfrak{h}$) induced by the canonical almost complex structures of $G_{\mathbb{R}}$ (resp. $H_{\mathbb{R}}$), then $d\phi \circ J_G = J_H \circ d\phi$.*

Now we establish the converse of Theorem 1.10.

Theorem 1.12 *Let G be a real analytic group with Lie algebra $\mathfrak{g}$, and assume that $\mathfrak{g}$ has a complex structure J. Then there is a unique complex analytic group $\widehat{G}$, whose underlying real analytic group is G, such that the given complex structure J is exactly the one induced from the canonical almost complex structure on $\widehat{G}$. In this case, the Lie algebra of $\widehat{G}$ is the complex Lie algebra $\widehat{\mathfrak{g}}$ associated with the complex structure J on $\mathfrak{g}$.*

Proof. First we prove the uniqueness. Suppose G is the underlying real analytic group of two complex analytic groups such that their canonical almost complex structures induce the same complex structure J. Then necessarily the two almost complex structures are the same, and hence by Corollary 1.5, the two complex analytic groups are identical, proving the uniqueness.

Recall that $\widehat{\mathfrak{g}}$ is the complex Lie algebra obtained from the real Lie algebra $\mathfrak{g}$ by extending the action of the base field $\mathbb{R}$ to $\mathbb{C}$ via

$$(a + \sqrt{-1}b)X = aX + bJ(X), \ a, \ b \in \mathbb{R}.$$

Choose an open neighborhood N of 0 in $\widehat{\mathfrak{g}}$ $(= \mathfrak{g})$ such that the Campbell-Hausdorff (C-H) multiplication $X \circ Y$ is defined for all $X, \ Y \in N$. We choose N small enough so that the complex analytic map $(X, Y) \mapsto X \circ Y$ defines a local complex analytic group structure on N and so that

$$\exp_G : N \to U = \exp_G(N)$$

is an isomorphism of local (real) analytic groups with its inverse $\log_G$ defined on U (see Theorem A.26). We transfer the local complex analytic group structure of N to U by means of $\exp_G$. It is evident that the underlying real analytic group structure of the local complex analytic group structure on U thus obtained coincides with the given local real analytic group structure on U. Our goal is to extend the local complex group structure just introduced on U to a complex analytic group structure of the entire G. Choose an open symmetric neighborhood V of 1 so that $V^2 \subset U$. Since U is a local complex analytic group, we obtain the following:

(i) The map $(x, y) \mapsto xy^{-1} : V \times V \to U$ is complex analytic;

(ii) Let A be an open subset of V and $v \in V$ so that $vA \subset V$. Then $L_v|_A : A \to vA$ is an isomorphism of complex analytic manifolds;

(iii) For each $x \in G$, the inner automorphism $I_x : y \mapsto xyx^{-1}$ is complex analytic, when restricted to a small neighborhood of 1 in V.

Now we introduce a complex analytic manifold structure on G. Choose an open symmetric neighborhood W of 1 in G such that $W^3 \subset V$. Define $\phi : U \to \widehat{\mathfrak{g}}$ by $\phi(x) = \log_G(x)$, and for each $x \in G$, define

$$\phi_x : xW \to \widehat{\mathfrak{g}}$$

by $\phi_x(z) = \phi(x^{-1}z),\ z \in xW$. Each pair (xW, ϕ_x) is an open chart at x, and the charts $(xW, \phi_x),\ x \in G$, define a complex analytic manifold structure on G. To see this, we must show that the charts are complex analytically compatible. Suppose $xW \cap yW$ is not empty. We want to show that

$$\phi_y \circ \phi_x^{-1} : \phi_x(xW \cap yW) \to \phi_y(xW \cap yW)$$

is complex analytic. We have $x^{-1}y,\ y^{-1}x \in W^2$, and if we put

$$D = \phi_x(xW \cap yW) = \phi(W \cap x^{-1}yW),$$

then D is an open subset of $N = \phi(U)$. For each $d \in D$, we have

$$\phi_y \circ \phi_x^{-1}(d) = \phi(y^{-1}x\phi^{-1}(d)),$$

and

$$\phi_y \circ \phi_x^{-1} = \phi \circ (L_{y^{-1}x}|_{W \cap x^{-1}yW}) \circ \phi^{-1}$$

follows. Since $L_{y^{-1}x}|_{W \cap x^{-1}yW}$ is complex analytic by (ii), we see that $\phi_y \circ \phi_x^{-1}$ is complex analytic.

From here on, the space G, equipped with the structure of the complex analytic manifold just introduced, is denoted by $\widehat{G}$. From the construction, it is evident that G is the underlying real analytic manifold of $\widehat{G}$. We also note that the left translation by each element of G leaves the atlas $\{(xW, \phi_x) : x \in G\}$ *invariant*, and hence we have:

(iv) For $z \in G$, the left translation $L_z : G \to G$ is complex analytic.

Now we are ready to show that the complex analytic manifold G is a complex analytic group by showing that the map $\eta : G \times G \to G$, given by $\eta(x, y) = xy^{-1}$, is complex analytic.

Let $(x_0, y_0) \in G \times G$. We may choose an open neighborhood V' of 1 in V such that $y_0 V' y_0^{-1} \subset V$ and that $I_{y_0} : z \mapsto y_0 z y_0^{-1}$ is complex analytic (by (iii)). Let W' be a symmetric open neighborhood of 1 such that $(W')^3 \subset V'$, and we show that η, when restricted to $x_0 W' \times y_0 W'$, is complex analytic. Note that, for $u, v \in W'$, we have

$$(x_0 u)(y_0 v)^{-1} = (x_0 y_0^{-1})(y_0(uv^{-1})y_0^{-1}) \in x_0 y_0^{-1} V,$$

and η hence maps $x_0 W' \times y_0 W'$ into $x_0 y_0^{-1} V$.

We consider the commutative diagram:

$$\begin{array}{ccc} W' \times W' & \xrightarrow{\ I_{y_0} \circ \eta\ } & V \\ \downarrow & & \downarrow \\ x_0 W' \times y_0 W' & \xrightarrow[\ \eta\ ]{} & x_0 y_0^{-1} V \end{array}$$

where the vertical maps are translations. Then we have

$$\eta|_{x_0 W' \times y_0 W'} = L_{x_0 y_0^{-1}} \circ I_{y_0} \circ (\eta|_{W' \times W'}) \circ (L_{x_0}^{-1} \times L_{y_0}^{-1}).$$

Since $L_{x_0 y_0^{-1}}$ and $L_{x_0}^{-1} \times L_{y_0}^{-1}$ are both complex analytic by (iv), and since $\eta|_{W' \times W'}$ is complex analytic by (i), it follows that

$$\eta : x_0 W' \times y_0 W' \longrightarrow x_0 y_0^{-1} V$$

is complex analytic.

It remains to show that J is exactly the complex structure on $\mathfrak{g}$ induced from the canonical almost complex analytic structure of $\widehat{G}$. Let J' denote the latter. Since G is the underlying real analytic group of $\widehat{G}$, we have $\mathcal{L}(\widehat{G})_{\mathbb{R}} = \mathfrak{g}$ by Proposition 1.9, and $J'(X) = \sqrt{-1}X$, $X \in \mathfrak{g}$, by Theorem 1.10, proving $J = J'$. ∎

It is clear from the proof of Theorem 1.12 that the canonical almost complex structure on G is obtained from the given complex structure J by transferring J to each tangent space $T_x(G)$. With this observation, we now present the converse of Theorem 1.11.

Theorem 1.13 *Let G and H be real analytic groups, and suppose that the Lie algebras of G and H are given with complex structures J and J', respectively. If $\phi : G \to H$ is a morphism of real analytic groups such that $d\phi \circ J = J' \circ d\phi$, then $\phi : \widehat{G} \to \widehat{H}$ is complex analytic.*

Proof. Let $\mathfrak{g}$ and $\mathfrak{h}$ denote the Lie algebras of G and H, respectively. Since $d\phi : \mathfrak{g} \to \mathfrak{h}$ commutes with the given complex structures, it commutes with the canonical almost complex structures of G and H, and hence the assertion follows from Proposition 1.4. ■

Suppose $\phi : G \to H$ is a morphism of real analytic groups, and assume that ϕ is surjective and that $\ker(\phi)$ is discrete. Then we have the Lie algebra isomorphism $d\phi : \mathfrak{g} \cong \mathfrak{h}$, and any complex structure on one Lie algebra can be transferred to that of the other by means of $d\phi$. Hence by Theorem 1.13, we see that H is the underlying real analytic group of a complex analytic group if and only if G is so, and, in this case, ϕ is a morphism of complex analytic groups. In particular we have

Corollary 1.14 (i) *The simply connected covering group of a complex analytic group admits the structure of a complex analytic group so that the covering map is a morphism of complex analytic groups.*

(ii) *The quotient group of a complex analytic group by a discrete subgroup admits the structure of a complex analytic group so that the quotient map is a morphism of complex analytic groups.* ■

An almost complex structure J on a differentiable manifold M is sometimes called a *complex structure*, if there is a complex analytic manifold $\widehat{M}$, whose underlying differentiable manifold is M, such that J is induced by the canonical almost complex structure of $\widehat{M}$. Using this terminology, Theorem 1.10 and Theorem 1.12 simply state that a complex structure of a real analytic group G induces a complex structure on $\mathcal{L}(G)$, and, conversely, every complex structure on $\mathcal{L}(G)$ is induced by a complex structure on G.

Complex Groups as Real Groups with Complex Structure Up to now, we have established that every complex analytic group may be viewed as a real analytic group whose Lie algebra has a complex structure, and conversely, every real analytic group whose Lie algebra is equipped with a complex structure may be given the structure of a complex analytic group that is compatible with the real analytic group structure of G. We may therefore characterize every complex analytic group as a real analytic group with complex structure on its Lie algebra.

We next characterize morphisms of complex analytic groups in terms of real analytic structures and the complex structures on their Lie algebras. Suppose G and K are complex analytic groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$, respectively. If $\phi : G \to K$ is a complex analytic homomorphism, its differential $d\phi$ is a morphism $\mathfrak{g} \to \mathfrak{k}$ of complex Lie algebras, and hence we may view ϕ as a real analytic homomorphism whose differential $d\phi$ commutes with the complex structures of $\mathfrak{g}$ and $\mathfrak{k}$. Conversely, suppose $\phi : G \to K$ is a morphism of real analytic groups such that $d\phi$ is $\mathbb{C}$-linear. By Theorem 1.13, ϕ is complex analytic.

We may summarize what has been discussed above as follows. When complex analytic groups are characterized as real analytic groups with complex structure on their Lie algebras, morphisms of complex analytic groups are exactly the morphisms of real analytic groups whose differentials are $\mathbb{C}$-linear.

Exponential Maps Suppose G is a complex analytic group with Lie algebra $\mathfrak{g}$. We have seen (in Proposition 1.9) that the Lie algebra of the underlying real analytic group $G_{\mathbb{R}}$ of G may be identified with the Lie algebra $\mathfrak{g}$, viewed as a real Lie algebra. Thus we have the exponential map of the real analytic group $G_{\mathbb{R}}$

$$\exp_{G_{\mathbb{R}}} : \mathfrak{g} \to G,$$

which is real analytic, and locally invertible at 0.

A complex analytic homomorphism $\mathbb{C} \to G$ is called a *one-parameter subgroup* in G . Noting that the differential operator $\frac{d}{dz}$ spans the Lie algebra of the additive group $\mathbb{C}$, each complex analytic 1-parameter subgroup ϕ in G determines an element $X \in \mathfrak{g}$, namely, $X = d\phi(\frac{d}{dz})$, and, conversely, given any $X \in \mathfrak{g}$ there is a unique complex analytic 1-parameter subgroup $\phi_X : \mathbb{C} \to G$ such that $d\phi_X(\frac{d}{dz}) = X$. In fact, we first identify the Lie algebra of the additive group $\mathbb{C}$ with $\mathbb{C}$ via the isomorphism

$$c \mapsto c\frac{d}{dz} : \mathbb{C} \to \mathcal{L}(\mathbb{C}).$$

We note that, under this identification, the condition $d\phi_X(\frac{d}{dz}) = X$ becomes $d\phi_X(1) = X$. Define the $\mathbb{C}$-linear map $\alpha : \mathbb{C} \to \mathfrak{g}$ by $\alpha(c) = cX,\ c \in \mathbb{C}$, and view it as a real (Lie algebra) homomorphism. Since the underlying real analytic group $\mathbb{C}_{\mathbb{R}}$ of $\mathbb{C}$ is simply connected, there exists a unique real analytic homomorphism

$$\phi_X : \mathbb{C} \to G$$

such that $d\phi_X = \alpha$, i.e., $\phi_X = \exp_{G_{\mathbb{R}}} \circ \alpha$. This shows that

$$\phi_X(c) = \exp_{G_{\mathbb{R}}}(cX),\ c \in \mathbb{C}. \tag{1.2.3}$$

Since $\alpha = d\phi_X$ is also a morphism of complex Lie algebras, ϕ_X is complex analytic, proving that it is the 1-parameter subgroup in G satisfying $d\phi_X(1) = X$.

We define the *exponential map* of the complex analytic group G,

$$\exp_G : \mathfrak{g} \to G,$$

by

$$\exp_G(X) = \phi_X(1),\ X \in \mathfrak{g}.$$

Theorem 1.15 *For a complex analytic group G with Lie algebra $\mathfrak{g}$, we have*

(i) $\exp_G = \exp_{G_{\mathbb{R}}}$;

(ii) $\exp_G$ *is a complex analytic map of* $\mathfrak{g}$ *into* G.

Proof. (i) For $X \in \mathfrak{g}$, we have, using (1.2.3),

$$\exp_G X = \phi_X(1) = \exp_{G_{\mathbb{R}}}(X),$$

proving $\exp_G = \exp_{G_{\mathbb{R}}}$.

(ii) Here $\mathfrak{g}$ is endowed with its natural structure as a complex analytic manifold which is isomorphic with $\mathbb{C}^n$, where $n = \dim_{\mathbb{C}} G$.

Choose an open connected neighborhood V of 0 in $\mathfrak{g}$ such that $\exp_{G_\mathbb{R}}$ is invertible, when restricted to V, and let $W = \exp_{G_\mathbb{R}}(V)$. Then we may view

$$(\exp_{G_\mathbb{R}} \mid V)^{-1} : W \to V$$

as a local chart of G at 1. Recalling that the (canonical) almost complex structure on G is obtained by transferring the natural almost complex structure on $\mathbb{C}^n$ by means of charts, we see immediately that the map $\exp_{G_\mathbb{R}}$ commutes with the almost structures on the open submanifolds V and W. Thus by Proposition 1.4, $\exp_G (= \exp_{G_\mathbb{R}})$ is complex analytic in an open neighborhood of 0 in $\mathfrak{g}$. To show that $\exp_G$ is complex analytic at every point in $\mathfrak{g}$, let $X \in \mathfrak{g}$. Then there is a positive integer m such that $m^{-1}X$ is contained in some open neighborhood of 0 on which $\exp_G$ is complex analytic. Since $\exp_G X = (\exp_G(m^{-1}X))^m$, we see that $\exp_G$ is complex analytic at X. ■

1.3 Examples of Complex Lie Groups

We collect some of the examples of basic nature for later use.

Complex Vector Groups Let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ (n-copies). Viewed as an additive group, it is a complex analytic group, and is the only simply connected abelian complex analytic group of dimension n. We call $\mathbb{C}^n$ (or any group isomorphic with it) simply a *complex vector group* (of dimension n). Let $z_1, \cdots, z_n$ be the natural coordinate system on $\mathbb{C}^n$. Then

$$\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}$$

form a basis for $\mathcal{L}(\mathbb{C}^n)$. Since $[\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}] = 0$ for all i, j, the Lie algebra $\mathcal{L}(\mathbb{C}^n)$ is abelian. We often identify $\mathcal{L}(\mathbb{C}^n)$ with the abelian Lie algebra $\mathbb{C}^n$. In that case, the exponential map for $\mathbb{C}^n$ is just the identity map.

Let $e_1, e_2, \cdots, e_{2n}$ be a basis of the $\mathbb{R}$-linear space $\mathbb{C}^n$, and let

$$D = \{\sum_{i=1}^{2n} m_i e_i : m_i \in \mathbb{Z}\}.$$

Then $\mathbb{C}^n/D$ is a compact complex analytic group. This is a real torus, and, surprisingly, this is the only kind of *compact* complex analytic group as we shall see in Theorem 1.19 below.

Remark 1.16 Although any two real tori of the same dimension are isomorphic (as real analytic groups), two compact (abelian) complex analytic groups of the same dimension may not be isomorphic as complex groups. To construct such groups, fix an irrational number α, and let D and D' be the discrete subgroups of the additive group $\mathbb{C}$ generated by $\{1, \sqrt{-1}\}$ and $\{1, \alpha\sqrt{-1}\}$, respectively. Then $\mathbb{C}/D$ and $\mathbb{C}/D'$ are nonisomorphic 1-dimensional compact complex analytic groups. In fact, any isomorphism between $\mathbb{C}/D$ and $\mathbb{C}/D'$ would induce an isomorphism $\phi : \mathbb{C} \to \mathbb{C}$ of complex Lie groups such that $\phi(D) \subset D'$. Then ϕ is $\mathbb{C}$-linear, and if we write

$$\phi(1) = m + n\alpha\sqrt{-1}$$

and

$$\phi(\sqrt{-1}) = p + q\alpha\sqrt{-1}$$

for some integers m, n, p, and q, then

$$p + q\alpha\sqrt{-1} = \phi(\sqrt{-1}) = \sqrt{-1}\phi(1) = -n\alpha + m\sqrt{-1}$$

yields $p = -n\alpha$ and $m = q\alpha$. This implies $p, q, m, n = 0$, which is impossible because ϕ is an isomorphism. ■

General Linear Lie Group Let $GL(n, \mathbb{C})$ denote the group of all nonsingular complex $n \times n$ matrices. We show that $GL(n, \mathbb{C})$ is a complex analytic group, and we determine its Lie algebra.

Let $M(n, \mathbb{C})$ denote the complex linear space consisting of all $n \times n$ complex matrices, and for $A \in M(n, \mathbb{C})$, let $x_{i,j}(A)$ denote the (i, j)-entry of A. We endow $M(n, \mathbb{C})$ with the structure of a complex analytic manifold so that the natural $\mathbb{C}$-linear isomorphism $M(n, \mathbb{C}) \to \mathbb{C}^{n^2}$ becomes an isomorphism of complex manifolds. The determinant map $\det : M(n, \mathbb{C}) \to \mathbb{C}$, being a polynomial in the coordinate functions $x_{i,j}$, is complex analytic, and thus $GL(n, \mathbb{C})$ is open in $M(n, \mathbb{C})$. We note that the space $GL(n, \mathbb{C})$ is topologically connected by Theorem 1.17, as we shall discuss below. Now the connected set $GL(n, \mathbb{C})$ may be viewed as an open submanifold of

$M(n,\mathbb{C}) = \mathbb{C}^{n^2}$. Equipped with this analytic structure, $GL(n,\mathbb{C})$ is a complex analytic group. In fact, the map

$$(a,b) \mapsto ab^{-1} : GL(n,\mathbb{C}) \times GL(n,\mathbb{C}) \to GL(n,\mathbb{C})$$

is complex analytic, because each $x_{i,j}(ab^{-1})$ is a rational function of the $x_{p,q}(a)$ and $x_{p,q}(b)$ whose denominator $\neq 0$ on $GL(n,\mathbb{C})$.

We determine $\mathcal{L}(GL(n,\mathbb{C}))$. First we let $\mathfrak{gl}(n,\mathbb{C})$ denote the Lie algebra of all complex $n \times n$ matrices, where the Lie algebra bracket $[\ ]$ is given by

$$[A,B] = AB - BA,\ \ A,B \in \mathfrak{gl}(n,\mathbb{C}).$$

For any $X \in \mathcal{L}(GL(n,\mathbb{C}))$, we denote by X' the matrix whose (i,j)-entry $X'_{i,j}$ is $X_1(x_{i,j})$. (Recall from §1.1 that, if X is a vector field on $GL(n,\mathbb{C})$, then X_1 denotes the corresponding tangent vector to $GL(n,\mathbb{C})$ at 1.)

The map

$$X \mapsto X' : \mathcal{L}(GL(n,\mathbb{C})) \to \mathfrak{gl}(n,\mathbb{C})$$

is clearly a $\mathbb{C}$-linear isomorphism. We now show that this map is an isomorphism of Lie algebras, i.e.,

$$[X,Y]' = [X',Y'],\ \ X,\ Y \in \mathcal{L}(GL(n,\mathbb{C})).$$

First note that, for $1 \leq i,\ j \leq n$, $Z \in \mathcal{L}(GL(n,\mathbb{C}))$ and $x \in G$, we have

$$x_{i,j} \circ L_x = \sum_k x_{i,k}(x)x_{k,j},$$

and, applying Z_1 to this, we have

$$Z_x(x_{i,j}) = dL_x(Z_1)(x_{i,j}) = Z_1(x_{i,j} \circ L_x) = \sum_k Z_1(x_{k,j})x_{i,k}$$

establishing the formula

$$Z_x(x_{i,j}) = \sum_{k=1}^{n} Z_1(x_{k,j})x_{i,k}(x). \tag{1.3.1}$$

Next, applying (1.3.1) to the vector fields X and Y,

$$(X \circ Y - Y \circ X)x_{i,j} = \sum_{k,l}(X_1(x_{l,k})Y_1(x_{k,j}) - X_1(x_{k,j})Y_1(x_{l,k}))x_{i,l},$$

and evaluating it at 1, we get

$$\begin{aligned}
[X,Y]'_{i,j} &= [X,Y]_1(x_{i,j}) \\
&= (X \circ Y - Y \circ X)_1(x_{i,j}) \\
&= \sum_k (X_1(x_{i,k})Y_1(x_{k,j}) - X_1(x_{k,j})Y_1(x_{i,k})) \\
&= \sum_k (X'_{i,k}Y'_{k,j} - Y'_{i,k}X'_{k,j}) \\
&= (X'Y')_{i,j} - (Y'X')_{i,j} \\
&= [X',Y']_{i,j},
\end{aligned}$$

proving $[X,Y]' = [X',Y']$.

If we identify the Lie algebra $\mathcal{L}(GL(n,\mathbb{C}))$ with $\mathfrak{gl}(n,\mathbb{C})$ using the isomorphism $X \mapsto X'$, then the exponential map

$$\exp : \mathfrak{gl}(n,\mathbb{C}) \to GL(n,\mathbb{C})$$

coincides with the usual one $X \mapsto e^X$, where

$$e^X = \sum_{i=0}^{\infty} \frac{X^i}{i!}, \ X \in \mathfrak{gl}(n,\mathbb{C}).$$

In fact, for each $X \in \mathcal{L}(GL(n,\mathbb{C}))$, there is a unique complex analytic 1-parameter subgroup $\phi_X : \mathbb{C} \to G$ such that $d\phi_X(\frac{d}{dz}) = X$. Let $z_0 \in \mathbb{C}$ and set $x_0 = \phi_X(z_0)$. Then $X_{x_0} = (d\phi_X)_{t_0}(\frac{d}{dz})_{t_0}$, and hence, for any complex analytic function f at x_0, we obtain

$$X_{x_0}(f) = (\frac{d(f \circ \phi_X)}{dz})_{z_0}.$$

In particular, we have

$$X_{x_0}(x_{i,j}) = (\frac{d(x_{i,j} \circ \phi_X)}{dz})_{z_0}. \qquad (1.3.2)$$

On the other hand, using the formula (1.3.1) applied to the vector field X, we deduce

$$X_{x_0}(x_{i,j}) = \sum_k x_{i,k}(x_0)X'_{k,j} = (x_0X')_{i,j} = (\phi_X(z_0)X')_{i,j}. \qquad (1.3.3)$$

From (1.3.2) and (1.3.3), it follows that $\phi_X(z)$ is a solution of the differential equation

$$\frac{dF(z)}{dz} = F(z)X',\ \ F(0) = 1.$$

Since this differential equation is also satisfied by $F(z) = e^{zX'}$, we see that $\phi_X(z) = e^{zX'}$ by the uniqueness of the solution, proving $\exp(X) = \phi_X(1) = e^{X'}$.

Let V be an n-dimensional $\mathbb{C}$-linear space, and we denote the group of all $\mathbb{C}$-linear automorphisms of V by $GL(V, \mathbb{C})$. Choose a basis $e_1, e_2, \cdots, e_n$ of V, and define $x_{i,j}(u) \in \mathbb{C}$ for $u \in GL(V, \mathbb{C})$ by the equation

$$u(e_j) = \sum_{i=1}^{n} x_{i,j}(u)e_i.$$

Then $u \mapsto (x_{i,j}(u))$ defines an isomorphism

$$\alpha_e : GL(V, \mathbb{C}) \to GL(n, \mathbb{C}).$$

We provide $GL(V, \mathbb{C})$ with the structure of a complex analytic Lie group by transferring the Lie group structure of $GL(n, \mathbb{C})$ to $GL(V, \mathbb{C})$ by means of α_e. That this complex Lie group structure on $GL(V, \mathbb{C})$ is independent of choice of basis of V may be seen as follows. Suppose $v_1, v_2, \cdots, v_n$ is another basis, and let $\alpha_v : GL(V, \mathbb{C}) \to GL(n, \mathbb{C})$ be the corresponding isomorphism. If we define $\theta \in GL(V, \mathbb{C})$ by $\theta(e_j) = v_j, 1 \leq j \leq n$, then

$$\alpha_e(u) = \alpha_v(\theta)\alpha_v(u)\alpha_v(\theta)^{-1}.$$

Since $x \mapsto \alpha_v(\theta)x\alpha_v(\theta)^{-1}$ is an automorphism of $GL(n, \mathbb{C})$, the two bases define the same analytic group structure on $GL(V, \mathbb{C})$.

Finally we note that, for a complex analytic group G with Lie algebra $\mathfrak{g}$, the adjoint representation $Ad : G \to GL(\mathfrak{g}, \mathbb{C})$ is complex analytic because its differential, the adjoint representation

$$ad : \mathfrak{g} \to \mathrm{gl}(\mathfrak{g}, \mathbb{C}),$$

is a morphism of complex Lie algebras.

Decomposition of $GL(V,\mathbb{C})$ Let V be a finite-dimensional linear space over $\mathbb{C}$, and let $<,>$ be a Hermitian inner product on V. For $x \in End_{\mathbb{C}}(V)$, let x^* denote the adjoint of x that is determined by the condition

$$< x(u), v >=< u, x^*(v) >$$

for all $u, v \in V$. $x \in End_{\mathbb{C}}(V)$ is called *Hermitian* (resp. *unitary*) if $x = x^*$ (resp. $xx^* = 1 = x^*x$). A Hermitian element $x \in End_{\mathbb{C}}(V)$ is called *positive definite* if $< x(u), u >$ is positive for all nonzero $u \in V$, or equivalently, if all the eigenvalues of x are positive. Let $\mathcal{S}(V)$ denote the set of all Hermitian elements in $End_{\mathbb{C}}(V)$, and let $\mathcal{P}(V)$ (resp. $\mathrm{U}(V)$) denote the set of all positive definite Hermitian elements (resp. the set of all unitary elements) in $GL(V,\mathbb{C})$. Then $\mathcal{S}(V)$ is an $\mathbb{R}$-linear subspace of $End_{\mathbb{C}}(V)$, and the usual exponential map

$$\exp : End_{\mathbb{C}}(V) \to GL(V,\mathbb{C})$$

maps $\mathcal{S}(V)$ homeomorphically onto $\mathcal{P}(V)$. We note that $\exp|_{\mathcal{S}(V)}$ is, in fact, an isomorphism of real analytic manifolds. See [4] (§IV-§V, Chapter I) for this discussion as well as the proof of the following decomposition theorem.

Theorem 1.17 *Let V be a finite-dimensional linear space over $\mathbb{C}$ together with a positive definite Hermitian from F on V. Then the multiplication map*

$$\psi : \mathcal{P}(V) \times \mathrm{U}(V) \to GL(V,\mathbb{C})$$

is an isomorphism of real analytic manifolds. ∎

In matrix language, Theorem 1.17 states that the analytic group $GL(n,\mathbb{C})$ has the decomposition

$$GL(n,\mathbb{C}) = \mathcal{P}(n) \times \mathrm{U}(n),$$

where $\mathrm{U}(n)$ is the group of all unitary matrices and $\mathcal{P}(n)$ denotes the subset of $GL(n,\mathbb{C})$ consisting of all positive definite Hermitian matrices. The subset $\mathcal{S}(n)$ consisting of all Hermitian matrices forms an $\mathbb{R}$-linear subspace of $M(n,\mathbb{C})$, and the usual exponential map $\exp : M(n,\mathbb{C}) \to GL(n,\mathbb{C})$ maps $\mathcal{S}(n)$ homeomorphically onto $\mathcal{P}(n)$.

The following theorem plays a crucial role in determining topological properties of some complex linear groups.

Theorem 1.18 *Let F be a positive definite Hermitian form on a finite-dimensional linear space V over $\mathbb{C}$ and let T be an algebraic subgroup of $GL(V, \mathbb{C})$. Assume that T is self-adjoint, that is, $x^* \in T$ whenever $x \in T$. Then the multiplication map*

$$\psi : (T \cap \mathcal{P}(V)) \times (T \cap \mathrm{U}(V)) \to T$$

is a homeomorphism. ■

Proof. In light of Theorem 1.17, it is enough to show that ψ is a bijection.
(A). We first show: If $x \in \mathcal{S}(V)$ with $\exp(x) \in T$, then $\exp \mathbb{C}x \subset T$. Since T is algebraic, it is sufficient to show that $P(\exp \mathbb{C}x) = 0$ for every polynomial function P on $End_{\mathbb{C}}(V)$ which vanishes on T. With respect to a suitably chosen basis of V, the Hermitian element x is represented by a diagonal matrix. Thus we may assume that x itself is a diagonal matrix, say

$$x = diag(a_1, \cdots, a_n)$$

with each $a_i \in \mathbb{C}$. Then $\exp(tx)$, $t \in \mathbb{C}$, is given by

$$\exp(tx) = diag(e^{ta_1}, \cdots, e^{ta_n}).$$

We show: $P(\exp(tx)) = 0$. Suppose $P(\exp(tx)) \neq 0$ for some $t \in \mathbb{C}$, and write $P(\exp(tx))$ as

$$P(\exp(tx)) = \sum_{i=1}^{q} b_i e^{tc_i},$$

where the b_i are all nonzero complex numbers and the c_i are all distinct. Now we put $s = \exp(x/2)$. Then $s^2 = \exp(x) \in T$, and hence $\exp(kx) = s^{2k} \in T$, $k = 0, 1, 2, \cdots$. Therefore we have

$$0 = P(s^{2k}) = P(\exp(kx)) = \sum_{i=1}^{q} b_i e^{kc_i},$$

for all k. Letting $e^{c_i} = d_i$, $1 \leq i \leq q$, we see that $b_1, \cdots, b_n$ is a solution of the system of equations

$$\sum_{i=1}^{q} d_i^j x_i = 0, \ 1 \leq j \leq q. \qquad (1.3.4)$$

On the other hand, since the d_i are all distinct, $\det(d_i^j) \neq 0$, and hence the system (1.3.4) has the unique solution $x_1 = 0, \cdots, x_n = 0$. Thus $b_1 = 0, \cdots, b_n = 0$, a contradiction, and this completes the proof of (A).

(B). (A) shows, in particular, that every element in $T \cap \mathcal{S}(V)$ is on a 1-parameter subgroup entirely lying on $T \cap \mathcal{S}(V)$, and hence $T \cap \mathcal{S}(V)$ is connected. Now take $z \in T$, and write $z = su$ with $s \in \mathcal{P}(V)$ and $u \in \mathrm{U}(V)$ (Theorem 1.17). Then $s^2 = zz^* \in T \cap \mathcal{S}(V)$, and this implies $s^2 = \exp(x)$ for some $x \in \mathcal{S}(V)$. $s = \exp(x/2) \in T \cap \mathcal{S}(V)$ by (A), and $u = s^{-1}z \in T \cap \mathrm{U}(V)$. This proves that ψ is surjective, and it is, in fact, bijective by Theorem 1.17. ■

Complex Tori Let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers. Thus $GL(1, \mathbb{C}) = \mathbb{C}^*$, and $\mathbb{C}^*$ becomes a complex analytic group with the analytic structure being that of an open submanifold of $\mathbb{C}$, and $\mathcal{L}(\mathbb{C}^*) = \mathfrak{gl}(1, \mathbb{C}) = \mathbb{C}^*$. For any integer $m \geq 1$, the product

$$(\mathbb{C}^*)^m = \mathbb{C}^* \times \cdots \times \mathbb{C}^* \ (m \text{ times})$$

is called a *complex torus*. This is a complex analytic group with Lie algebra $\mathbb{C}^m$. Let $D(n, \mathbb{C})$ denote the subgroup of $GL(n, \mathbb{C})$ consisting of all invertible diagonal matrices. Then $D(n, \mathbb{C}) \cong (\mathbb{C}^*)^n$, and $\mathcal{L}(D(n, \mathbb{C})) = \mathfrak{d}(n, \mathbb{C})$, the Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ consisting of all diagonal matrices.

Theorem 1.19 *Every compact complex analytic group G is abelian, and hence is a (real) torus.*

Proof. We first note that a complex analytic function on a compact manifold must be a constant (see [3], VI.4.4, p. 192). The adjoint representation $Ad : G \to GL(\mathcal{L}(G), \mathbb{C})$, being a complex analytic map, is therefore constant. This shows that $Ad(G) = 0$, and G is abelian. For the second assertion, the universal covering group of G is isomorphic with $\mathbb{C}^n$, where $n = \dim_{\mathbb{C}}(G)$, and hence $G \cong \mathbb{C}^n/D$, where D is some discrete subgroup of $\mathbb{C}^n$. ■

Remark 1.20 It is well known that a real torus admits a faithful linear representation. However as we shall see later (Corollary 4.5), a nontrivial compact complex analytic group never admits a faithful complex analytic linear representation. ■

Some Complex Linear Lie Groups Let det : $GL(n, \mathbb{C}) \to \mathbb{C}^*$ denote the determinant map, and let $Tr : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$ denote the trace map.

Lemma 1.21 *(i)* $\det(e^u) = e^{Tr(u)}$, $u \in \mathfrak{gl}(n, \mathbb{C})$*;*

(ii) $d(\det) = Tr$.

Proof. If u is an upper triangular matrix, then so is e^u, and hence the assertion (i) follows. Otherwise, u is similar to an upper triangular matrix, i.e., there exist $x \in GL(n, \mathbb{C})$ and a triangular matrix a such that $u = xax^{-1}$. Then

$$\begin{aligned} \det(e^u) &= \det e^{xax^{-1}} = \det(xe^a x^{-1}) = \det(e^a) \\ &= e^{Tr(a)} = e^{Tr(xax^{-1})} = e^{Tr(u)}, \end{aligned}$$

proving (i).

To prove (ii), let $u \in \mathfrak{gl}(n, \mathbb{C})$. Then

$$\begin{aligned} d(\det)(u) &= d(\det)(\frac{de^{zu}}{dz} |_{z=0}) = \frac{d(\det(e^{zu}))}{dz} |_{z=0} \\ &= \frac{de^{zTr(u)}}{dz} |_{z=0} = Tr(u). \end{aligned}$$

■

Below we list some of the basic complex linear Lie subgroups of $GL(n, \mathbb{C})$.

(a) The subgroup $T(n, \mathbb{C})$ of $GL(n, \mathbb{C})$ that consists of all upper triangular matrices is a complex analytic subgroup of $GL(n, \mathbb{C})$, and $\mathcal{L}(T(n, \mathbb{C})) = \mathfrak{t}(n, \mathbb{C})$, the complex Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ of all upper triangular matrices.

(b) The subgroup $\mathrm{U}(n, \mathbb{C})$ of $GL(n, \mathbb{C})$ of all upper triangular matrices with all diagonal entries 1 is a complex analytic subgroup of $GL(n, \mathbb{C})$, and $\mathcal{L}(\mathrm{U}(n, \mathbb{C})) = \mathfrak{n}(n, \mathbb{C})$, the complex subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ of all strictly upper triangular matrices. $\mathrm{U}(n, \mathbb{C})$ is nilpotent and the exponential map $exp : \mathrm{n}(n, \mathbb{C}) \to \mathrm{U}(n, \mathbb{C})$ is given by

$$X \mapsto e^X = \sum_{i=0}^{n-1} \frac{X^i}{i!},$$

and is an isomorphism of complex analytic manifolds. Hence $\mathrm{U}(n, \mathbb{C})$ is, in particular, simply connected.

(c) Let $SL(n,\mathbb{C})$ denote the subgroup of $GL(n,\mathbb{C})$ that consists of all matrices x with $\det(x) = 1$. This is a closed subgroup of $GL(n,\mathbb{C})$, known as the *special linear group*, and is topologically connected. Indeed, $SL(n,\mathfrak{C})$ is a self-adjoint algebraic subgroup of $GL(n,\mathfrak{C})$, and hence by Theorem 1.18, every $u \in SL(n,\mathbb{C})$ can be written uniquely as $u = kp$, where $k \in SU(n)$ and p is a positive definite Hermitian matrix with $\det(p) = 1$, and this decomposition provides a homeomorphism

$$SL(n,\mathbb{C}) = SU(n) \times \mathbb{R}^{n^2-1},$$

proving, in particular, that $SL(n,\mathbb{C})$ is connected.

Now $SL(n,\mathbb{C})$ is a complex Lie subgroup of $GL(n,\mathbb{C})$. To see this, it is enough to observe that the Lie algebra of $SL(n,\mathfrak{C})$ is the complex subalgebra $\mathfrak{sl}(n,\mathbb{C})$, which consists of all matrices $u \in \mathfrak{gl}(n,\mathbb{C})$ with $Tr(u) = 0$. In fact, $SL(n,\mathbb{C}) = \ker(\det)$ shows that it is a closed real analytic subgroup of $GL(n,\mathbb{C})$, and, using Lemma 1.21, we obtain

$$\mathcal{L}(SL(n,\mathbb{C})) = \mathcal{L}(\ker(\det)) = \ker(d\det) = \ker(Tr) = \mathfrak{sl}(n,\mathbb{C}).$$

Normalizer and Centralizer For a subgroup H of a group G, the normalizer $N_G(H)$ of H in G,

$$N_G(H) = \{g \in G : gHg^{-1} = H\},$$

is a subgroup of G. Suppose now that G is a complex analytic group with Lie algebra $\mathfrak{g}$. If M is a complex linear subspace of $\mathfrak{g}$, then

$$N_G(M) = \{g \in G : Ad(g)(M) \subset M\}$$

is a real Lie subgroup as a closed subgroup of G. The Lie algebra of $N_G(M)$ is easily seen to be

$$N_{\mathfrak{g}}(M) = \{u \in \mathfrak{g} : ad(u)(M) \subset M\},$$

which is a complex Lie subalgebra of $\mathfrak{g}$, and $N_G(M)$ is therefore a complex Lie subgroup of G.

Proposition 1.22 *If H is a complex analytic subgroup of a complex analytic group G, then $N_G(H)$ is a closed complex Lie subgroup of G.*

Proof. Let $\mathfrak{g} = \mathcal{L}(G)$, and $\mathfrak{h} = \mathcal{L}(H)$. Then $N_G(H) = N_G(\mathfrak{h})$, and the assertion follows from the above discussion. ■

Now we study the centralizer of a subset of a complex Lie group. We begin with the following general result.

Proposition 1.23 *Let G be a complex Lie group G, and suppose G acts complex analytically on a complex analytic manifold M under the action*

$$(g, y) \mapsto g \cdot y : G \times M \to M.$$

Then the stabilizer subgroup at any $x \in M$,

$$G_x = \{g \in G : g \cdot x = x\},$$

is a closed complex Lie subgroup of G.

Proof. Clearly G_x is closed in G, and hence is a real Lie subgroup of G. Therefore it is enough to show that its Lie algebra $\mathcal{L}(G_x)$ is a complex linear space. Let $u \in \mathcal{L}(G_x)$. Then we have

$$\exp tu \in G_x \ \forall\, t \in \mathbb{R}. \tag{1.3.5}$$

Choose a local chart (U, ϕ) at x in M so that $\phi(x) = (0, \cdots, 0) \in \mathbb{C}^n$, $n = \dim(M)$, and consider the function

$$f(c) = \phi((\exp cu) \cdot x),$$

which is defined and complex analytic in a neighborhood of 0 in $\mathbb{C}$ with values in $\mathbb{C}^n$. Then $f(t) = 0$ for all *real* t by (1.3.5). Since the set of zeros of any nonzero complex analytic function on a connected open set in $\mathbb{C}$ is discrete (see, e.g., [3], Prop. 4.1., p. 41), we see that $f(c) = 0$ for *all complex numbers* c with $|c|$ small. This implies $cu \in \mathcal{L}(G_x)$ for all $c \in \mathbb{C}$, proving that $\mathcal{L}(G_x)$ is a $\mathbb{C}$-linear space. ■

A complex analytic group G acts on itself by conjugation, and hence by Proposition 1.23, the centralizer G_x of $x \in G$ is a closed complex analytic subgroup. As an immediate consequence, we have

Proposition 1.24 *If X is a subset of a complex analytic group G, then the centralizer of X in G,*

$$Z_G(X) = \{u \in G : uxu^{-1} = x \ \forall x \in X\},$$

is a closed complex Lie subgroup of G. In particular, the center of G is a closed complex Lie subgroup. ■

1.4 The Automorphism Group

Automorphisms and Derivations Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$. The (Lie algebra) automorphisms of $\mathfrak{g}$ form a subgroup of the general linear group $GL(\mathfrak{g}, \mathbb{F})$. We denote this subgroup by $Aut(\mathfrak{g})$. In the case $\mathbb{F} = \mathbb{R}$, $Aut(\mathfrak{g})$ is a Lie subgroup of $GL(\mathfrak{g}, \mathbb{R})$, and its Lie algebra is $Der(\mathfrak{g})$ ([4], Proposition 1, p. 137). On the other hand, if $\mathfrak{g}$ is a complex Lie algebra, $Aut(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g}, \mathbb{C})$, and in fact we have

$$Aut(\mathfrak{g}) = Aut(\mathfrak{g}_{\mathbb{R}}) \cap GL(\mathfrak{g}, \mathbb{C}).$$

Proposition 1.25 *For a finite-dimensional complex Lie algebra $\mathfrak{g}$, $Aut(\mathfrak{g})$ is a complex Lie group, and its Lie algebra is $Der(\mathfrak{g})$.*

Proof. As before, $\mathfrak{g}_{\mathbb{R}}$ denotes the underlying real Lie algebra of $\mathfrak{g}$. Then $\mathcal{L}(Aut(\mathfrak{g}_{\mathbb{R}})) = Der(\mathfrak{g}_{\mathbb{R}})$. $Aut(\mathfrak{g})$ is a real Lie group as a closed subgroup of $GL(\mathfrak{g}, \mathbb{C})$, and we have $Aut(\mathfrak{g}) = Aut(\mathfrak{g}_{\mathbb{R}}) \cap GL(\mathfrak{g}, \mathbb{C})$. The proof of the first assertion of the proposition amounts to showing that the Lie algebra $\mathcal{L}(Aut(\mathfrak{g}))$ of $Aut(\mathfrak{g})$ is a *complex* Lie subalgebra of $\mathfrak{gl}(\mathfrak{g}, \mathbb{C})$. Thus the first assertion follows as soon as we have shown the second assertion: $\mathcal{L}(Aut(\mathfrak{g})) = Der(\mathfrak{g})$. We first show

$$\mathcal{L}(Aut(\mathfrak{g})) \subset Der(\mathfrak{g}).$$

Since $Der(\mathfrak{g}) = Der(\mathfrak{g}_{\mathbb{R}}) \cap \mathfrak{gl}(\mathfrak{g}, \mathbb{C})$ and $\mathcal{L}(Aut(\mathfrak{g}))$ is a subalgebra of the real Lie algebra $\mathcal{L}(Aut(\mathfrak{g}_{\mathbb{R}})) = Der(\mathfrak{g}_{\mathbb{R}})$, it is enough to show that every element of $\mathcal{L}(Aut(\mathfrak{g}))$ is $\mathbb{C}$-linear. For $\delta \in \mathcal{L}(Aut(\mathfrak{g}))$ and $t \in \mathbb{R}$, we have

$$e^{t\delta} \in Aut(\mathfrak{g}).$$

Thus $e^{t\delta} : \mathfrak{g} \to \mathfrak{g}$ is $\mathbb{C}$-linear. For any $X \in \mathfrak{g}$ and $c \in \mathbb{C}$, we then have

$$e^{t\delta}(cX) = ce^{t\delta}(X), \ t \in \mathbb{R},$$

and differentiating both sides of the above equation with respect to t at 0, we obtain

$$\delta(cX) = c\delta(X).$$

This proves that δ is $\mathbb{C}$-linear.

We next show $Der(\mathfrak{g}) \subset \mathcal{L}(Aut(\mathfrak{g}))$. Let $\delta \in Der(\mathfrak{g})$, and let $t \in \mathbb{R}$. From the inclusion

$$Der(\mathfrak{g}) \subset Der(\mathfrak{g}_{\mathbb{R}}) = \mathcal{L}(Aut(\mathfrak{g}_{\mathbb{R}})),$$

it follows
$$e^{t\delta} \in Aut(\mathfrak{g}_{\mathbb{R}}). \tag{1.4.1}$$
On the other hand,
$$\delta \in Der(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g}, \mathbb{C}) = \mathcal{L}(GL(\mathfrak{g}, \mathbb{C}))$$
implies
$$e^{t\delta} \in GL(\mathfrak{g}, \mathbb{C}). \tag{1.4.2}$$
Thus (1.4.1) and (1.4.2) yield
$$e^{t\delta} \in Aut(\mathfrak{g}_{\mathbb{R}}) \cap GL(\mathfrak{g}, \mathbb{C}) = Aut(\mathfrak{g}),$$
and $\delta \in \mathcal{L}(Aut(\mathfrak{g}))$ follows. This shows $Der(\mathfrak{g}) \subset \mathcal{L}(Aut(\mathfrak{g}))$, and we have proved $Der(\mathfrak{g}) = \mathcal{L}(Aut(\mathfrak{g}))$. ■

Automorphism Groups Given a complex analytic group G, let $Aut(G)$ denote the group of all (complex analytic) automorphisms of G. We want to introduce the structure of a Lie group on $Aut(G)$. We first note that the canonical injective homomorphism
$$\alpha \mapsto d\alpha : Aut(G) \to Aut(\mathfrak{g})$$
is an isomorphism if G is simply connected. Let $\widetilde{G}$ be the universal covering group of G so that $G = \widetilde{G}/D$ for some discrete normal subgroup D of $\widetilde{G}$. Noting that $\widetilde{G}$ is a complex analytic group (Corollary 1.14) and that $Aut(\mathfrak{g})$ is a complex Lie group (Proposition 1.25), we may equip the group $Aut(\widetilde{G})$ with the structure of a complex Lie group so that the natural isomorphism
$$Aut(\widetilde{G}) \to Aut(\mathfrak{g})$$
becomes an isomorphism of complex Lie groups. On the other hand, the group $Aut(G)$ may be viewed as the closed subgroup of $Aut(\widetilde{G})$ consisting of all $\alpha \in Aut(\widetilde{G})$ such that $\alpha(D) \subset D$. In particular, $Aut(G)$ is a real Lie subgroup of the complex Lie group $Aut(\widetilde{G})$. The canonical map
$$Aut(\widetilde{G}) \times \widetilde{G} \to \widetilde{G}$$
is continuous, when the groups are viewed as real Lie groups. We retain the notation introduced above to prove the following:

Proposition 1.26 *$Aut(G)$ is a complex Lie subgroup of $Aut(\widetilde{G})$.*

Proof. The natural action of $Aut(\widetilde{G})$ on $\widetilde{G}$ is complex analytic, and hence by Proposition 1.23, the stabilizer subgroup at any $z \in \widetilde{G}$,

$$A_z = \{\alpha \in Aut(\widetilde{G}) : \alpha(z) = z\},$$

is a closed complex Lie subgroup of $Aut(\widetilde{G})$. Let $Aut_0(G)$ denote the identity component of $Aut(G)$. The natural action $Aut(G) \times G \to G$ maps the set $Aut_0(G) \times D$ into D. Since D is discrete, the action maps $Aut_0(G) \times \{z\}$ into $\{z\}$, i.e., $Aut_0(G) \subset A_z$ for each $z \in D$. Consequently we have

$$Aut_0(G) \subset \cap_{z \in D} A_z \subset Aut(G).$$

Thus the complex subgroup $\cap_{z \in D} A_z$ is an open subgroup of $Aut(G)$, and consequently, $Aut(G)$ is a complex Lie group. ■

As a closed complex Lie subgroup of $GL(\mathfrak{g}, \mathbb{C})$, the natural action of $Aut(\mathfrak{g})$ on $\mathfrak{g}$ is linear, and hence it is complex analytic. We use this fact to prove

Proposition 1.27 *If G is a complex analytic group, then the natural action*

$$\phi : Aut(G) \times G \to G,$$

mapping (α, x) to $\alpha(x)$, is complex analytic.

Proof. Let G be a complex analytic group with Lie algebra $\mathfrak{g}$, and let $\widetilde{G}$ denote the simply connected covering group of G. Then $Aut(G)$ is a closed complex Lie subgroup of $Aut(\widetilde{G})$ by Proposition 1.26. Thus our assertion follows as soon as we have shown the assertion for the simply connected $\widetilde{G}$, and hence we may assume that G itself is simply connected. Also every neighborhood of the identity element $1 \in G$ generates G, and ϕ is easily seen to be complex analytic in each variable separately. Therefore it is enough to show the following: For a fixed $\alpha_0 \in Aut(G)$, ϕ is complex analytic at $(\alpha_0, 1) \in Aut(\mathfrak{g}) \times \mathfrak{g}$. The natural action of $Aut(\mathfrak{g})$ on $\mathfrak{g}$,

$$\phi' : Aut(\mathfrak{g}) \times \mathfrak{g} \to \mathfrak{g},$$

is linear, and hence it is clearly complex analytic. Choose an open neighborhood $\mathcal{B}$ of 0 in $\mathfrak{g}$ so that $\exp_G$ is invertible on $\mathcal{B}$ (i.e., $\log_G$ is defined on $\exp(\mathcal{B})$). Let $W = \exp(B)$, and let U be an open symmetric neighborhood of 1 in G so that W contains both $\overline{U}$ and $\overline{\alpha_0(U)}$. Then there exists an open neighborhood $\mathcal{A}$ of $d\alpha_0 \in Aut(\mathfrak{g})$ such that $\phi'(\mathcal{A} \times \log_G(U)) \subset \mathcal{B}$. If V is the inverse image of $\mathcal{A}$ under the map $\alpha \mapsto d\alpha : Aut(G) \to Aut(\mathfrak{g})$, then $\phi : (\alpha, x) \mapsto \alpha(x)$ is equal to the composite of the complex analytic maps $(\alpha, x) \mapsto (d\alpha, \log_G x)$, ϕ' and $\exp_G$ on $V \times U$. This proves that ϕ is analytic on $V \times U$. ■

Semidirect Products Let H and N be complex analytic groups and let

$$\phi : H \to Aut(N)$$

be a morphism of complex Lie groups. The *semidirect product* of N by H, denoted by $N \rtimes_\phi H$ (or simply $N \rtimes H$, if ϕ is well understood) is the topological group defined on the product manifold $N \times H$, for which the multiplication is given by

$$(n_1, h_1) \cdot (n_2, h_2) = (n_1 \phi(h_1)(n_2), h_1 h_2).$$

By Proposition 1.27, the action of H on N,

$$(h, n) \mapsto \phi(h)(n) : H \times N \to N,$$

is complex analytic, and it now becomes easy to see that the map

$$((n_1, h_1), (n_2, h_2)) \mapsto (n_1, h_1) \cdot (n_2, h_2)^{-1} = (n_1 \phi(h_1^{-1})(n_2^{-1}), h_1 h_2^{-1})$$

is complex analytic. Consequently, the semidirect product $N \rtimes_\phi H$ acquires the structure of a complex analytic group.

To determine the Lie algebra of $N \rtimes_\phi H$, let $\mathfrak{n}$ and $\mathfrak{h}$ be the Lie algebras of N and H, respectively, and identify $Aut(N)$ with a closed complex Lie subgroup of the complex Lie group $Aut(\mathfrak{n})$ via the natural injection $Aut(N) \to Aut(\mathfrak{n})$. Then the differential $d\phi$ is a Lie algebra homomorphism

$$d\phi : \mathfrak{h} \to Der(\mathfrak{n}),$$

and we may form the semidirect sum $\mathfrak{n} \oplus_{d\phi} \mathfrak{h}$ (see §A.1). We have

Proposition 1.28 *There is a canonical isomorphism of complex Lie algebras*

$$\mathcal{L}(N \rtimes_\phi H) \cong \mathfrak{n} \oplus_{d\phi} \mathfrak{h}.$$

Proof. Let $\pi : N \rtimes H \to H$ be the projection, and let $\iota : N \to N \rtimes H$ and $\sigma : H \to N \rtimes H$ denote the canonical injections. Then $\pi \circ \sigma = 1_H$ implies $d\pi \circ d\sigma = 1_{\mathfrak{h}}$, and the map

$$F : \mathfrak{n} \oplus_{d\phi} \mathfrak{h} \to \mathcal{L}(N \rtimes_\phi H),$$

sending (α, γ) to $d\iota(\alpha) + d\sigma(\gamma)$, is easily seen to be an injective $\mathbb{C}$-linear map, and the comparison of dimension shows that it is an isomorphism. We show that F is a morphism of Lie algebras. The Lie algebra morphism condition

$$F([(\alpha, \gamma),\ (\beta, \eta)]) = [F(\alpha, \gamma),\ F(\beta, \eta)],$$

for $(\alpha, \gamma),\ (\beta, \eta) \in \mathfrak{n} \oplus_{d\phi} \mathfrak{h}$, is equivalent to the condition

$$d\iota \circ d\phi(\gamma) = ad(d\sigma(\gamma) \circ d\iota),\ \ \gamma \in \mathfrak{h}. \tag{1.4.3}$$

Here ad denotes the adjoint representation of $\mathcal{L}(N \times_\phi H)$. We now prove (1.4.3) below. Let exp denote the exponential map for $N \rtimes_\phi H$. Then we have

(i) $e^{ad(X)} = Ad(\exp X),\ \ X \in \mathcal{L}(N \rtimes_\phi H)$;

(ii) $I_{\sigma(y)} \circ \iota = \iota \circ \phi(y),\ \ y \in H$;

(iii) $\phi \circ \exp_H(\gamma) = e^{d\phi(\gamma)},\ \ \gamma \in \mathfrak{h}$,

and, using these identities, we obtain

$$e^{t\, ad(d\sigma(\gamma))} \circ d\iota = d\iota \circ e^{t\, d\phi(\gamma)},$$

for all $t \in \mathbb{R}$. Expanding and equating the coefficients of t in the above equation, we get (1.4.3). ■

1.5 Universal Complexification

We introduce an important construction of a complex analytic group from a real analytic group, known as the universal complexification. By a *universal complexification* of a real analytic group G, we mean a pair (G^+, γ), where G^+ is a complex analytic group, and $\gamma : G \to G^+$ is a morphism of real Lie groups satisfying the following universal property. For any complex analytic group K and any morphism $u : G \to K$ (of real Lie groups), there exists a unique complex analytic morphism $u^+ : G^+ \to K$ such that $u = u^+ \circ \gamma$. The pair (G^+, γ) is determined by G uniquely up to an isomorphism.

Construction of Universal Complexification Throughout the construction of (G^+, γ), we use the following notation. For any finite-dimensional real or complex Lie algebra $\mathfrak{g}$, let $S(\mathfrak{g})$ denote the simply connected real or complex analytic group whose Lie algebra is $\mathfrak{g}$.

Let G be a real analytic group with Lie algebra $\mathfrak{g}$. There exists a morphism

$$h : S(\mathfrak{g}) \to S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}) \tag{1.5.1}$$

of real Lie groups (necessarily a local injection) whose differential is the canonical injection $\mathfrak{g} \to \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$. Let F denote the kernel of the universal covering morphism $\pi : S(\mathfrak{g}) \to G$. For any real analytic homomorphism $u : G \to K$, where K is a complex analytic group, the differential $du : \mathfrak{g} \to \mathcal{L}(K)$ extends canonically to a complex Lie algebra morphism $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g} \to \mathcal{L}(K)$, and the latter is the differential of a morphism $u^* : S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}) \to K$ of complex Lie groups. We note that u^* is uniquely determined by u subject to the commutativity of the diagram:

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\;h\;} & S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}) \\ \pi \downarrow & & \downarrow u^* \\ G & \xrightarrow[\;u\;]{} & K \end{array} \tag{1.5.2}$$

Let N be the intersection of all kernels $\ker(u^*)$, where u ranges over the real analytic homomorphisms $u : G \to K$ (with varying K). Since $\mathcal{L}(N)$ is the intersection of the ideals $\mathcal{L}(\ker(u^*))$, $\mathcal{L}(N)$ is an ideal of the complex Lie algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$, and hence the quotient group $S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g})/N$ is a complex analytic group. We denote this group by G^+. From $u^* \circ h = u \circ \pi$, it follows that $h(F) \subset \ker(u^*)$, and hence we have $h(F) \subset N$. Let $\gamma : G \to G^+$ be the morphism that is induced from h by passing to the quotients. We show that the pair (G^+, γ) is a universal complexification. For any real analytic homomorphism $u : G \to K$, where K is a complex analytic group, $u^* : S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}) \to K$ maps N to 1, and hence induces a complex analytic morphism $u^+ : G^+ \to K$ such that $u^* = u^+ \circ \pi_c$, where $\pi_c : S(\mathbb{C} \otimes \mathfrak{g}) \to G^+$ denotes the quotient map. We have

$$(u^+ \circ \gamma) \circ \pi = u^+ \circ \pi_c \circ h = u^* \circ h = u \circ \pi,$$

and this implies $u^+ \circ \gamma = u$. Now it remains to show the uniqueness of u^+. Suppose there is a complex analytic morphism $u' : G^+ \to K$

such that $u' \circ \gamma = u$. From $u' \circ \gamma = u^+ \circ \gamma$, it follows $u' = u^+$ on $Im(\gamma)$, and hence $du' = du^+$ on $Im(d\gamma)$. Since $Im(\gamma)$ spans $\mathcal{L}(G^+)$ over $\mathbb{C}$, $du' = du^+$, and $u' = u^+$ follows.

Remark 1.29 For later use, we assemble some additional facts from the construction of G^+.

(i) The image of $S(\mathfrak{g})$ under the map h in (1.5.1) is closed in $S(\mathbb{C} \otimes \mathfrak{g})$, and hence the map $h : S(\mathfrak{g}) \to h(S(\mathfrak{g}))$ is a covering morphism of the closed real analytic subgroup $h(S(\mathfrak{g}))$ of $S(\mathbb{C} \otimes \mathfrak{g})$.

In fact, the complex conjugation of the complexification

$$X + \sqrt{-1}Y \mapsto X - \sqrt{-1}Y : \mathbb{C} \otimes \mathfrak{g} \to \mathbb{C} \otimes \mathfrak{g}, \ X, \ Y \in \mathfrak{g}$$

is the differential of a real analytic involution θ of $S(\mathbb{C} \otimes \mathfrak{g})$, and $h(S(\mathfrak{g}))$ is closed as it is the identity component of the subgroup of $S(\mathbb{C} \otimes \mathfrak{g})$ consisting of all fixed points of θ.

(ii) The subgroup $h(F)$ is central in $S(\mathbb{C} \otimes \mathfrak{g})$. This can be seen as follows: h maps the discrete (and hence central) subgroup F of $S(\mathfrak{g})$ onto a central subgroup of $h(S(\mathfrak{g}))$, and the centralizer Z of $h(F)$ in $S(\mathbb{C} \otimes \mathfrak{g})$ therefore contains $h(S(\mathfrak{g}))$. But Z is a closed complex Lie subgroup of $S(\mathbb{C} \otimes \mathfrak{g})$ (Proposition 1.24), and $S(\mathbb{C} \otimes \mathfrak{g})$ is the smallest complex analytic subgroup that contains $h(S(\mathfrak{g}))$. Thus $Z = S(\mathbb{C} \otimes \mathfrak{g})$, proving the assertion.

(iii) The normal subgroup N that appears in the construction of G^+ above is a central subgroup of $S(\mathbb{C} \otimes \mathfrak{g})$ and is equal to the smallest closed complex Lie subgroup that contains $h(F)$. To see this, let M be the smallest closed complex Lie subgroup of $S(\mathbb{C} \otimes \mathfrak{g})$ that contains $h(F)$. Then M is central in $S(\mathbb{C} \otimes \mathfrak{g})$ by (ii) above, and is contained in N. We claim $M = N$ by proving $N \subset M$. Let $K = S(\mathbb{C} \otimes \mathfrak{g})/M$, and let

$$t : G = S(\mathfrak{g})/F \to K = S(\mathbb{C} \otimes \mathfrak{g})/M$$

denote the morphism induced by h. As seen in the construction of G^+ above (see the diagram (1.5.2)), t induces a unique morphism

$$t^* : S(\mathbb{C} \otimes \mathfrak{g}) \to K$$

so that we have a commutative diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\ h\ } & S(\mathbb{C} \otimes \mathfrak{g}) \\ \pi \downarrow & & \downarrow t^* \\ G & \xrightarrow[\ t\ ]{} & K \end{array} \qquad (1.5.3)$$

Since the diagram (1.5.3) remains commutative when t^* is replaced by the canonical complex analytic morphism

$$\sigma : S(\mathbb{C} \otimes \mathfrak{g}) \to K = S(\mathbb{C} \otimes \mathfrak{g})/M,$$

the uniqueness of t^* yields $t^* = \sigma$, and we have

$$N \subset \ker(t^*) = \ker(\sigma) = M.$$

This proves $M = N$.

(iv) In general, it is difficult to determine the subgroup N. In the case where the subgroup $h(F)$ itself is discrete, $h(F)$ is trivially a complex Lie subgroup, and $h(F) = N$ by (iii). In this case, we have $G^+ = S(\mathbb{C} \otimes \mathfrak{g})/h(F)$. Shortly, we shall examine some of the important classes of groups for which $h(F)$ is discrete. ■

Below we determine some sufficient conditions for real analytic groups having injective canonical maps. First, we consider real analytic groups having continuous faithful representations. For this we need the following definition.

A real Lie subalgebra $\mathfrak{g}_0$ of a complex Lie algebra $\mathfrak{g}$ is called a *real form* of $\mathfrak{g}$, if the canonical map $\mathbb{C} \otimes \mathfrak{g}_0 \to \mathfrak{g}$ is an isomorphism of complex Lie algebras, or equivalently, if $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$.

Proposition 1.30 *Let G be a real analytic group with Lie algebra $\mathfrak{g}$, and suppose that G admits a finite dimensional faithful continuous representation. Then the canonical map $\gamma : G \to G^+$ is injective, and the Lie algebra $\mathcal{L}(\gamma(G))(= d\gamma(\mathfrak{g}))$ is a real form of $\mathcal{L}(G^+)$.* ■

Proof. Let $\eta : G \to GL(W, \mathbb{R})$ be a faithful analytic representation on a finite-dimensional real linear space W. Embedding $GL(W, \mathbb{R})$ into the general linear group $GL(\mathbb{C} \otimes_{\mathbb{R}} W, \mathbb{C})$, we may assume that there is an *injective* morphism $u : G \to K$, where K is a complex analytic group. It follows from $u = u^+ \circ \gamma$ and from the injectivity of u that γ is an injection.

We also have the commutative diagram (1.5.2), in which u is now an injection, and from this we obtain $\ker(h) \subset F$. By Remark 1.29 (i), the map $S(\mathfrak{g}) \to h(S(\mathfrak{g}))$ is a covering morphism of analytic groups, and hence $h(F)(\cong F/\ker h)$ is a discrete central subgroup of $S(\mathbb{C} \otimes \mathfrak{g})$, and $G^+ = S(\mathbb{C} \otimes \mathfrak{g})/h(F)$ by Remark 1.29 (iv). If we identify $\mathcal{L}(G^+)$ with $\mathbb{C} \otimes \mathfrak{g}$, then the differential $d\gamma : \mathcal{L}(G) \to \mathcal{L}(G^+)$ is the canonical injection $\mathfrak{g} \longrightarrow \mathbb{C} \otimes \mathfrak{g}$, and hence $Im(d\gamma)$ is a real form of $\mathcal{L}(G^+)$. ∎

We next consider the universal complexification of solvable analytic groups. We recall that every analytic subgroup of a simply connected solvable analytic group is closed in G and is simply connected. Using this, we now prove

Proposition 1.31 *If G is any real solvable analytic group, then the canonical map $\gamma : G \to G^+$ is injective.*

Proof. Since G is solvable, so are $S(\mathfrak{g})$, $S(\mathbb{C} \otimes \mathfrak{g})$, and the image $Im(h)$ of

$$h : S(\mathfrak{g}) \to S(\mathbb{C} \otimes \mathfrak{g})$$

is simply connected as a closed analytic subgroup of the simply connected solvable group $S(\mathbb{C} \otimes \mathfrak{g})$. Thus we have an isomorphism $h : S(\mathfrak{g}) \cong Im(h)$, and $h(F)$ is therefore a discrete (hence central) subgroup of $S(\mathbb{C} \otimes \mathfrak{g})$. By Remark 1.29(iv), $G^+ = S(\mathbb{C} \otimes \mathfrak{g})/h(F)$, and γ is injective. ∎

As Proposition 1.31 indicates, the converse of Proposition 1.30 is not valid in general. For a morphism $\phi : H \to G$ of real analytic groups, it is easy to show that $\phi^+ : H^+ \to G^+$ is surjective if ϕ is so. However, the injective ϕ does not always give the injective ϕ^+. Here is an example.

Example 1.32 Let

$$\sigma : \widetilde{SL}(2, \mathbb{R}) \to SL(2, \mathbb{R})$$

be the universal covering of $SL(2, \mathbb{R})$. The kernel of σ is an infinite cyclic group, and choose a generator a for $\ker(\sigma)$. Let $H = \mathbb{R}/\mathbb{Z}$ and let

$$G = \widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}/D,$$

where D is the subgroup of $\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R}$:

$$D = \{(a^n, \sqrt{2}n + m) : n, m \in \mathbb{Z}\}.$$

The morphism $\phi : H \to G$, given by

$$\phi(u + \mathbb{Z}) = (1, u)D, u \in \mathbb{R},$$

is injective. We claim that ϕ^+ is *not* injective. We first note that $H^+ = \mathbb{C}/\mathbb{Z}$ and that the canonical map γ_H is the usual inclusion $\mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}/\mathbb{Z}$. Next, we determine G^+. We have

$$\mathcal{L}(G) = \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R};$$

and

$$\mathbb{C} \otimes \mathcal{L}(G) = \mathfrak{sl}(2, \mathbb{C}) \times \mathbb{C}.$$

The canonical injection

$$\mathcal{L}(G) = \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R} \to \mathbb{C} \otimes \mathcal{L}(G) = \mathfrak{sl}(2, \mathbb{C}) \times \mathbb{C}$$

is the differential of the morphism

$$h : \widetilde{SL}(2, \mathbb{R}) \times \mathbb{R} \xrightarrow{h} SL(2, \mathbb{C}) \times \mathbb{C},$$

given by $h(x, r) = (\sigma(x), r)$. We have the following commutative diagram

$$\begin{CD}
\widetilde{SL}(2, \mathbb{R}) \times \mathbb{R} @>{h}>> SL(2, \mathbb{C}) \times \mathbb{C} \\
@V{\pi}VV @VVV \\
G @>>{\gamma_G}> G^+
\end{CD}$$

It is easy to see that $h(D) = \{1\} \times (\sqrt{2}\mathbb{Z} + \mathbb{Z})$, and the smallest closed complex Lie subgroup N of $SL(2, \mathbb{C}) \times \mathbb{C}$ that contains $h(D)$ is $\{1\} \times \mathbb{C}$, because $\sqrt{2}\mathbb{Z} + \mathbb{Z}$ is dense in $\mathbb{R}$. This shows

$$G^+ = (SL(2, \mathbb{C}) \times \mathbb{C})/N = SL(2, \mathbb{C}).$$

Now the canonical map γ_G maps $Im(\phi) = (\{1\} \times \mathbb{R})D/D$ to N so that $\gamma_G \circ \phi(\mathbb{R}/\mathbb{Z}) = 1$. This, together with the commutativity of the diagram,

$$\begin{CD}
\mathbb{R}/\mathbb{Z} @>{\theta}>> G \\
@VVV @VV{\gamma_G}V \\
\mathbb{C}^* @>>{\theta^+}> G^+ = SL(2, \mathbb{C})
\end{CD}$$

shows that ϕ^+ cannot be injective. ■

Using a result in Remark 1.29, we prove the following, which will be used in Chapter 4 (see the proof of Theorem 4.29).

Proposition 1.33 *Let G_1 and G_2 be compact real analytic groups, and suppose $\sigma : G_1 \to G_2$ is a covering morphism. Then the induced morphism $\sigma^+ : G_1^+ \to G_2^+$ is a covering morphism of complex analytic groups with finite kernel.*

Proof. We identify $\mathcal{L}(G_1)$ with $\mathcal{L}(G_2)$, and denote it by $\mathfrak{g}$. Using the notation in the construction of the universal complexification, let

$$h : S(\mathfrak{g}) \to S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g})$$

denote the real analytic homomorphism whose differential is the canonical injection $\mathfrak{g} \to \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$, and let $\pi_i : S(\mathfrak{g}) \to G_i$ be the universal covering morphism of G_i with its kernel F_i, $(i = 1, 2)$. From $\sigma \circ \pi_1 = \pi_2$, we have $F_1 \subset F_2$. Since the compact group G_i $(i = 1, 2)$ has a continuous finite-dimensional faithful representation, $h(F_i)$ is discrete (see the proof of Proposition 1.30), and we have (Remark 1.29 (iv))

$$G_i^+ = S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g})/h(F_i), \ i = 1, 2.$$

Thus $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$ may be identified with the Lie algebra of both G_1^+ and G_2^+. In this case, σ^+ is exactly the induced map

$$G_1^+ = S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g})/h(F_1) \to G_2^+ = S(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g})/h(F_2).$$

σ^+ is thus a covering morphism with kernel $h(F_2)/h(F_1)$. Since G_1 is compact, $\ker(\sigma) = F_2/F_1$ is finite, and hence $\ker(\sigma^+) = h(F_2)/h(F_1)$ is finite. ■

Universal Complexification of Semidirect Products Let G and H be real analytic groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, and let $\phi : H \to Aut(G)$ be a morphism of real Lie groups. There is a unique morphism $\phi' : H \longrightarrow Aut(G^+)$ defined by the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \phi(h)\ } & G \\ \downarrow & & \downarrow \\ G^+ & \xrightarrow[\ \phi'(h)\ ]{} & G^+ \end{array}$$

and ϕ', in turn, induces a morphism

$$\phi^+ : H^+ \to Aut(G^+)$$

of complex Lie groups making the diagram

$$\begin{array}{ccc} H & & \\ \downarrow & \searrow^{\phi'} & \\ H^+ & \xrightarrow[\phi^+]{} & Aut(G^+) \end{array}$$

commutative. We now form the semidirect products $G \rtimes_\phi H$ and $G^+ \rtimes_{\phi^+} H^+$ with respect to the actions ϕ and ϕ^+, respectively.

Proposition 1.34 *There is a canonical isomorphism*

$$(G \rtimes_\phi H)^+ \cong G^+ \rtimes_{\phi^+} H^+.$$

Proof. Let $\mathfrak{g}^+$ and $\mathfrak{h}^+$ denote the Lie algebras of G^+ and H^+, respectively. The map

$$\gamma : G \rtimes_\phi H \to G^+ \rtimes_{\phi^+} H^+$$

defined by $\gamma = \gamma_G \times \gamma_H$ (i.e., $\gamma(g, h) = (\gamma_G(g), \gamma_H(h))$) is easily seen to be a morphism of real Lie groups. We claim that $G^+ \rtimes_{\phi^+} H^+$, together with γ, satisfies the universal property characterizing the universal complexification of $G \rtimes_\phi H$. Let $\alpha : G \rtimes_\phi H \to M$ be a real morphism into a complex analytic group M, and define

$$\alpha^+ : G^+ \rtimes_{\phi^+} H^+ \to M$$

by $\alpha^+(g', h') = \alpha_G^+(g')\alpha_H^+(h')$. We show that α^+ is a homomorphism. The homomorphism condition

$$\alpha^+((g'_1, h'_1)(g'_2, h'_2)) = \alpha^+(g'_1, h'_1)\alpha^+(g'_2, h'_2),$$

for $(g'_1, h'_1),\ (g'_2, h'_2) \in G^+ \rtimes H^+$, is equivalent to the condition

$$\alpha_G^+(\phi^+(h')(g')) = \alpha_H^+(h')\alpha_G^+(g')\alpha_H^+(h')^{-1} \tag{1.5.4}$$

for $h' \in H^+$ and $g' \in G^+$.

Since $\alpha : G \rtimes_\phi H \to M$ is a homomorphism, we have

$$\alpha_G(\phi(h)(g)) = \alpha_H(h)\alpha_G(g)\alpha_H(h)^{-1}. \tag{1.5.5}$$

Define $\widehat{\phi^+} : H^+ \times G^+ \to G^+$ and $\widehat{\phi} : H \times G \to G$ by

$$\widehat{\phi^+}(h', g') = \phi^+(h')(g'); \ \widehat{\phi}(h, g) = \phi(h)(g).$$

Then the equations (1.5.4) and (1.5.5) are, respectively, equivalent to the commutativity of the diagrams:

$$\begin{array}{ccc} H^+ \times G^+ & \xrightarrow{\widehat{\theta^+}} & G^+ \\ {\scriptstyle \alpha_H^+ \times \alpha_G^+} \downarrow & & \downarrow {\scriptstyle \alpha_G^+} \\ M \times M & \xrightarrow[\eta]{} & M \end{array} \tag{1.5.6}$$

and

$$\begin{array}{ccc} H \times G & \xrightarrow{\hat{\phi}} & G \\ {\scriptstyle \alpha_H \times \alpha_G} \downarrow & & \downarrow {\scriptstyle \alpha_G} \\ M \times M & \xrightarrow[\eta]{} & M \end{array} \tag{1.5.7}$$

where $\eta : M \times M \to M$ is given by $\eta(x, y) = xyx^{-1}$. The diagrams (1.5.6) and (1.5.7) are commutative if and only if their respective Lie algebra diagrams (1.5.8) and (1.5.9)

$$\begin{array}{ccc} \mathfrak{h}^+ \times \mathfrak{g}^+ & \xrightarrow{\widehat{d\phi^+}} & \mathfrak{g}^+ \\ {\scriptstyle d\alpha_H^+ \times d\alpha_G^+} \downarrow & & \downarrow {\scriptstyle d\alpha_G^+} \\ \mathfrak{m} \times \mathfrak{m} & \xrightarrow[d\eta]{} & \mathfrak{m} \end{array} \tag{1.5.8}$$

and

$$\begin{array}{ccc} \mathfrak{h} \times \mathfrak{g} & \xrightarrow{\widehat{d\phi}} & \mathfrak{g} \\ {\scriptstyle d\alpha_H \times d\alpha_G} \downarrow & & \downarrow {\scriptstyle d\alpha_G} \\ \mathfrak{m} \times \mathfrak{m} & \xrightarrow[d\eta]{} & \mathfrak{m} \end{array} \tag{1.5.9}$$

are commutative. Thus to establish the equation (1.5.4), it is enough to show that the diagram (1.5.8) is commutative. The commutativity of (1.5.7),

$$\alpha_G \circ \widehat{\phi} = \eta \circ (\alpha_H \times \alpha_G),$$

implies

$$\alpha_G^+ \circ \widehat{\phi^+} = \eta \circ (\alpha_H^+ \times \alpha_G^+)$$

on $Im(\gamma)$, and hence we have

$$d\alpha_G^+ \circ d\widehat{\phi^+} = d\eta \circ (d\alpha_H^+ \times d\alpha_G^+) \tag{1.5.10}$$

on $Im(d\gamma)$. But $Im(d\gamma)(= Im(d\gamma_H \times d\gamma_G))$ spans $\mathfrak{h}^+ \oplus \mathfrak{g}^+$ over $\mathbb{C}$, and the differential of $\widehat{\phi^+}$

$$d\widehat{\phi^+} : \mathfrak{h}^+ \times \mathfrak{g}^+ \longrightarrow \mathfrak{g}^+$$

is a $\mathbb{C}$-linear action of the complex Lie algebra $\mathfrak{h}^+$ on $\mathfrak{g}^+$. Thus (1.5.10) provides

$$d\alpha_G^+ \circ d\widehat{\phi^+} = d\eta \circ (d\alpha_H^+ \times d\alpha_G^+),$$

proving that the diagram (1.5.8) is commutative.

That α^+ is the only morphism with the relation $\alpha = \alpha^+ \circ \gamma$ follows from the fact that $\mathrm{Im}(d\gamma)$ spans $\mathcal{L}(G^+ \rtimes_{\phi^+} H^+) = \mathfrak{g}^+ \oplus_{d\phi^+} \mathfrak{h}^+$ over $\mathbb{C}$. ■

Chapter 2

Representative Functions of Lie Groups

In this chapter we present the theory of representative functions, which arises from the study of representations of Lie groups. With this we introduce a pro-affine algebraic group associated with each Lie group, known as the universal algebraic hull. This links up the Lie group theory with the theory of pro-affine algebraic groups ([12], [16], [25]).

Throughout this chapter, $\mathbb{F}$ will denote a field of characteristic 0.

2.1 Basic Definitions

We assemble basic definitions and results on group representations for later use. Let V be a linear space over a field $\mathbb{F}$. Given a group G, a homomorphism $\rho : G \to GL(V, \mathbb{F})$ is called a *representation* of G on V, and V is called the *representation space* of ρ. If $\dim_{\mathbb{F}}(V)$ is finite, then ρ is said to be *finite-dimensional*, and the $\dim_{\mathbb{F}}(V)$ is called the *degree* of ρ. Representations of a group we deal with in this book are assumed to be finite-dimensional, unless stated otherwise. By an *action* of a group G on a set S, we mean a map

$$(x, s) \mapsto xs : G \times S \to S$$

satisfying $1s = s$, $s \in V$, and $x(ys) = (xy)s$, $x, y \in G$ and $s \in S$. An action of a group G on an $\mathbb{F}$-linear space V is called *linear* if

$x(\lambda u + v) = \lambda(xu) + xv$ for all $x \in G$, $u, v \in V$, and $\lambda \in \mathbb{F}$. A finite-dimensional $\mathbb{F}$-linear space on which the group G acts linearly is called a *G-module* (over $\mathbb{F}$). Given a representation ρ of G, its representation space V may be viewed as a G-module, where G acts on V by $xv = \rho(x)(v)$, for $x \in G$ and $v \in V$. Conversely, a G-module V is the representation space of a representation ρ by setting $\rho(x)(v) = xv$, $x \in G$ and $v \in V$. This correspondence enables us to view each representation of a group G as a G-module, and conversely. If V is a G-module with corresponding representation ρ, the representation of G that corresponds to any sub G-module W of the G-module V is called a *subrepresentation* of ρ.

If V and W are G-modules, then the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are also G-modules in a natural way; latter, for example, via the action

$$x(u \otimes v) = xu \otimes xv.$$

If ρ and σ are the representations of G corresponding to the G-modules V and W, respectively, then the representations corresponding to the direct sum $V \oplus W$ and the tensor product $V \otimes W$ are denoted by $\rho \oplus \sigma$ and $\rho \otimes \sigma$, respectively.

Dual Representation Given a representation $\rho : G \to GL(V, \mathbb{F})$ of a group G, the representation $\rho^\circ : G \to GL(V^*, \mathbb{F})$ of G on the dual space V^* of V, which is given by the equation

$$\rho^\circ(x)(\lambda)(v) = \lambda(\rho(x^{-1})(v)),\ x \in G,\ \lambda \in V^*,\ v \in V,$$

is called the *dual representation* of ρ. The associated G-module V^* is called the *dual module* of the G-module V.

Now if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the general linear group $GL(V, \mathbb{F})$ has the natural (real or complex) Lie group structure, and the analyticity of a representation ρ of a Lie group is with respect to this analytic structure on $GL(V, \mathbb{F})$. Thus ρ is $\mathbb{F}$-analytic if and only if all of its coefficient functions are $\mathbb{F}$-analytic. If ρ is an analytic representation of a real or complex Lie group, then so is ρ°.

Semisimple Representations A representation $\rho : G \to GL(V, \mathbb{F})$ of G is called *semisimple* if each G-stable subspace of V has a complementary G-stable subspace. In this case, V is called a *semisimple*

G-module. The representation ρ is called *irreducible* if V and (0) are the only G-stable subspaces of V.

Since we are under the assumption that the base field $\mathbb{F}$ is of characteristic 0, we have the following result. (For a proof, see, e.g., [4], Proposition 2, p. 88.)

Theorem 2.1 *If $\rho_i : G \to GL(V_i, \mathbb{F})$, $(i = 1, 2)$, are semisimple representations, then so is the tensor product $\rho_1 \otimes \rho_2$.* ■

For any representation $\rho : G \to GL(V, \mathbb{F})$, let

$$\Sigma : (0) = V_{m+1} \subset V_m \subset \cdots \subset V_1 \subset V_0 = V$$

be a series of sub G-modules of the G-module V. The G-module, which is the direct sum

$$V(\Sigma) = \oplus_{i=0}^{m} (V_i / V_{i+1})$$

of the factor G-modules V_i/V_{i+1}, provides in a canonical way a new representation

$$\rho^{\Sigma} : G \to GL(V(\Sigma), \mathbb{F}).$$

The representation ρ^{Σ} is called the *representation associated with ρ* (*relative to the series* Σ). If the series Σ is a composition series, then the G-module $V(\Sigma)$ is semisimple, and is simply denoted by V'. The corresponding representation

$$\rho' : G \to GL(V', \mathbb{F})$$

is called the *semisimple representation associated with ρ.* Note that because of the uniqueness of composition series for a G-module, ρ' is uniquely determined (up to equivalence) by ρ. On the other hand, if ρ itself is semisimple, then ρ and ρ' are *equivalent* in the sense that there is an F-linear isomorphism $V \to V'$ such that, for each $x \in G$, we have the commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\rho(x)} & V \\ \downarrow & & \downarrow \\ V' & \xrightarrow[\rho'(x)]{} & V' \end{array}$$

More generally, the following may be shown easily.

Lemma 2.2 *If a representation $\rho : G \to GL(V, \mathbb{F})$ is semisimple, then, for* any *series Σ of sub G-modules of the G-module V, ρ^{Σ} is also semisimple, and ρ and ρ^{Σ} are equivalent.* ■

Unipotent Representations Given a finite dimensional linear space V over a field $\mathbb{F}$, an $\mathbb{F}$-linear endomorphism u of V is called *unipotent* if $u-1$ is nilpotent, or equivalently, if its characteristic polynomial is $(X-1)^n$, where $n = \dim_{\mathbb{F}} V$. A subgroup of $GL(V,\mathbb{F})$ is said to be *unipotent* if all of its elements are unipotent. Similarly, $x \in GL(n,\mathbb{F})$ is *unipotent* if $x-1$ is nilpotent, and a subgroup of $GL(n,\mathbb{F})$ is called *unipotent* if all of its elements are unipotent. The subgroup $U(n,\mathbb{F})$ of $GL(n,\mathbb{F})$ consisting of all $n \times n$ upper triangular matrices with diagonal entries all 1 is clearly unipotent, and in fact $U(n,\mathbb{F})$ consists of all unipotent elements of $T(n,\mathbb{F})$, since the eigenvalues of the characteristic polynomial of a triangular matrix are the diagonal entries.

A representation $\rho : G \to GL(V,\mathbb{F})$ is called *unipotent* if the set of endomorphisms $\rho(x) - 1_V$, $x \in G$, is nilpotent on V. (Recall that a subset $S \subset End(V,\mathbb{F})$ is called *nilpotent on* V if there is a positive integer n such that the product of every sequence of n elements of S is 0.) As we shall see in Corollary 2.5, the unipotency of ρ is equivalent to the statement that $\rho(G)$ is a unipotent subgroup.

Proposition 2.3 *Let G be a unipotent subgroup of $GL(V,\mathbb{F})$. Then there is a nonzero $v \in V$ such that $x(v) = v$ for all $x \in G$.*

Proof. First we reduce the proposition to the case where the field $\mathbb{F}$ is algebraically closed. Let $\mathbb{K}$ be an algebraically closed field that contains $\mathbb{F}$ as a subfield, and let $V^{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{F}} V$. For each $x \in GL(V,\mathbb{F})$,

$$1_{\mathbb{K}} \otimes x : V^{\mathbb{K}} \to V^{\mathbb{K}}$$

is a $\mathbb{K}$-linear isomorphism of $V^{\mathbb{K}}$, and hence

$$G^{\mathbb{K}} = \{1_{\mathbb{K}} \otimes x : x \in GL(V,\mathbb{F})\}$$

is a subgroup of $GL(V^{\mathbb{K}},\mathbb{K})$. Then the groups G and $G^{\mathbb{K}}$ share the properties described in the proposition, and thus replacing G by $G^{\mathbb{K}}$, if necessary, we may assume that $\mathbb{F}$ itself is algebraically closed.

Suppose V has a proper nonzero G-stable subspace W. Then the image of G in $GL(W,\mathbb{F})$ is unipotent, and a common eigenvector of this group in W is also a common eigenvector of G in V. Thus we may further assume that V is simple as a G-module (i.e., V contains no proper G-stable subspace). In this case we shall show $G = (1)$.

Let $S = \{x - 1 : x \in G\}$ and let E be the $\mathbb{F}$-linear subspace of $End(V, \mathbb{F})$ spanned by S. Noting that, for $x,\ y \in G$,

$$(x-1)(y-1) = (xy-1) - (x-1) - (y-1) \in E,$$

we see that E is closed under the product of matrices, i.e., E is a subalgebra of $End(V, \mathbb{F})$. Since V is simple as a G-module, it is also simple as an E-module, and consequently, if $E \neq (0)$, then $E = End(V, \mathbb{F})$ by a theorem of Burnside (see, e.g., p. 648 of Lang's *Algebra*, 3rd ed.). On the other hand, since the set S consists of nilpotent elements by assumption, every element of S (and hence of E) has trace 0. But this is impossible because $E = End(V, \mathbb{F})$. This shows that $E = (0)$, proving the proposition. ■

Suppose $G \subset GL(V, \mathbb{F})$ is unipotent. Then G has a common eigenvector $v_1 \in V$ by Proposition 2.3. Letting $V_1 = \mathbb{F}v_1$, G induces a unipotent subgroup of $GL(V/V_1, \mathbb{F})$. Thus by induction applied to $\dim V$, we find a full flag of subspaces of V stable under G, i.e., a series of G-stable subspaces of V

$$V_1 \subset V_2 \subset \cdots \subset V_n = V,\ n = \dim V,$$

each properly contained in the next, such that the action of G on each factor V_{i+1}/V_i, $1 \leq i \leq n$, is trivial. Putting the above discussion in matrix language, we have

Theorem 2.4 *Let G be a subgroup of $GL(n, \mathbb{F})$. Then G is unipotent if and only if it is conjugate to a subgroup of $U(n, \mathbb{F})$. In particular, every unipotent subgroup is nilpotent.* ■

Corollary 2.5 *For a representation $\rho : G \to GL(V, \mathbb{F})$, the following are equivalent.*

(i) *ρ is unipotent.*

(ii) *$\rho(G)$ is a unipotent subgroup of $GL(V, \mathbb{F})$.*

(iii) *The semisimple representation ρ' associated with ρ is trivial.*

Proof. (i)⇒ (ii) is trivial.

(ii)⇒ (iii): Under the hypothesis (ii), there is a composition series for the G-module V, namely, a full flag of $\rho(G)$-stable subspaces

$$V_1 \subset V_2 \subset \cdots \subset V_n = V, \tag{2.1.1}$$

such that the induced action of G on each factor V_{i+1}/V_i is trivial. It follows from this that the semisimple representation ρ' associated with ρ is trivial.

(iii) $\Rightarrow$ (i): If ρ' is trivial, then clearly there is a $\rho(G)$-stable full flag (2.1.1) of subspaces of V for which the induced action of G on each factor V_{i+1}/V_i is trivial. Therefore, for any $x \in G$, we have $(\rho(x) - 1)(V_i) \subset V_{i-1}$, $1 \leq i \leq n$, where $V_0 = (0)$, and hence (i) follows. ∎

Closed Sets of Representations Let G be a complex (resp. real) analytic group, and let $\mathrm{Rep}(G)$ denote the set of all complex (resp. real) analytic representations of G. We say that a nonempty subset $\mathcal{E}$ of $\mathrm{Rep}(G)$ is said to be *closed* if it satisfies the following conditions:

(i) If $\rho, \sigma \in \mathcal{E}$, then $\rho \oplus \sigma$, $\rho \otimes \sigma \in \mathcal{E}$;

(ii) If ρ is an irreducible subrepresentation of $\sigma \in \mathcal{E}$, then $\rho \in \mathcal{E}$;

(iii) If $\sigma \in \mathcal{E}$ and if ρ is equivalent to σ, then $\rho \in \mathcal{E}$.

Given a subset $\mathcal{E}$ of $\mathrm{Rep}(G)$, there exists the smallest closed set in $\mathrm{Rep}(G)$ containing $\mathcal{E}$, namely, the intersection of all closed sets that contain $\mathcal{E}$. This is called the *set generated by* $\mathcal{E}$, and is denoted by $[\mathcal{E}]$. In the case G is an analytic group having the property that every analytic representation of G is semisimple, then we may construct $[\mathcal{E}]$ explicitly from $\mathcal{E}$ as follows. Let $\mathcal{E}_1$ denote the set of all irreducible subrepresentations of all finite tensor products

$$\rho_1 \otimes \cdots \otimes \rho_m$$

where $\rho_i \in \mathcal{E}$, $1 \leq i \leq m$. Note that if σ_1, $\sigma_2 \in \mathcal{E}_1$, then every irreducible subrepresentation of $\sigma_1 \otimes \sigma_2$ belongs to $\mathcal{E}_1$. Hence we see that the closed set $[\mathcal{E}]$ is exactly the set of all representations in $\mathrm{Rep}(G)$ which are equivalent to the sums

$$\sigma_1 \oplus \cdots \oplus \sigma_k$$

where $\sigma_j \in \mathcal{E}_1$, $1 \leq j \leq k$.

Example 2.6 For later use we present three contrasting examples of faithful complex analytic representations of the vector group $\mathbb{C}^n$. For the sake of simplicity, all representations are given in matrix form.

(i)

$$(x_1, \cdots, x_n) \overset{\rho}{\mapsto} \begin{pmatrix} e^{x_1} & 0 & \cdot & \cdot & 0 & 0 \\ 0 & e^{ix_1} & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & e^{x_n} & 0 \\ 0 & \cdot & \cdot & \cdot & 0 & e^{ix_n} \end{pmatrix}$$

is a faithful *semisimple* analytic representation of $\mathbb{C}^n$.

(ii)

$$(x_1, \cdots, x_n) \overset{\sigma}{\mapsto} \begin{pmatrix} 1 & 0 & \cdot & 0 & x_1 \\ 0 & 1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & x_n \\ 0 & \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

is a faithful analytic *unipotent* representation of the complex vector group $\mathbb{C}^n$.

(iii)

$$(x_1, \cdots, x_n) \overset{\varphi}{\mapsto} \begin{pmatrix} e^{x_1} & & 0 & & & & \\ & \cdot & & & & 0 & \\ 0 & & e^{x_n} & & & & \\ & & & 1 & & & x_1 \\ & & & & \cdot & & \cdot \\ & 0 & & & & \cdot & x_n \\ & & & & 0 & & 1 \end{pmatrix}$$

is a faithful representation, and $\mathrm{Im}(\varphi)$ is neither of the above types.

■

2.2 Algebras of Representative Functions

For a field $\mathbb{F}$ of characteristic 0, and a nonempty set X, let $\mathbb{F}^X$ denote the $\mathbb{F}$-algebra of all functions $X \to \mathbb{F}$.

Representative Functions Let G be a group. For a function $f : G \to \mathbb{F}$ and $x \in G$, we define the *left translate* $x \cdot f$ and the *right translate* $f \cdot x$ by $x \cdot f(y) = f(yx)$ and $f \cdot x(y) = f(xy)$, $y \in G$. The function f is called a *representative function* if all its translates lie in a finite-dimensional subspace of $\mathbb{F}^G$. This condition is easily seen to be equivalent to the assertion that its left translates $G \cdot f$ (equivalently, the right translates $f \cdot G$) span a finite-dimensional linear subspace of $\mathbb{F}^G$. The latter condition also shows that the representative functions on G form an $\mathbb{F}$-algebra under the usual addition and multiplication of functions, which is denoted by $R_{\mathbb{F}}(G)$. $R_{\mathbb{F}}(G)$ is stable under left translations and right translations by the elements of G.

The following general lemma concerning linear spaces of functions is useful in the study of representative functions.

Lemma 2.7 *Let X be a nonempty set. If V is a nonzero finite-dimensional subspace of $\mathbb{F}^X$, then there exists a basis $f_1, \cdots, f_n$ for V and elements $x_1, \cdots, x_n \in X$ such that $f_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq n$.*

Proof. Let V^* denote the dual space of V, and define $\mu : X \to V^*$ by $\mu(x)(f) = f(x)$ for $x \in X$ and $f \in V$. Then V^* is spanned by $\mu(X)$. In fact, let T be the subspace of V^* spanned by $\mu(X)$, and consider the dual pairing

$$(\lambda, f) \mapsto <\lambda, f> = \lambda(f) : V^* \times V \to \mathbb{F}.$$

We have

$$T^* \cong V/Ann(T),$$

where $Ann(S)$ for any subset $S \subset V^*$ denotes the subset of $V = V^{**}$

$$Ann(S) = \{f \in V : <\lambda, f> = 0 \text{ for all } \lambda \in S\}.$$

But $Ann(T) = Ann(\mu(X))$, and since $Ann(\mu(X)) = 0$, we have $T^* \cong V$, and $T = V^*$ follows. Choose elements $x_1, \cdots, x_n \in X$ such that $\mu(x_1), \cdots, \mu(x_n)$ form a basis for V^*, and let $f_1, \cdots, f_n$ be the basis for $V^{**} = V$, which is dual to the basis $\mu(x_1), \cdots, \mu(x_n)$. Then $f_i(x_j) = <\mu(x_j), f_i> = \delta_{ij}$ for $1 \leq i, j \leq n$. ■

We now introduce a comultiplication on $R_{\mathbb{F}}(G)$,

$$\gamma : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(G) \otimes R_{\mathbb{F}}(G).$$

Let

$$\pi : R_{\mathbb{F}}(G) \otimes R_{\mathbb{F}}(G) \to \mathbb{F}^{G\times G}$$

be the $\mathbb{F}$-linear map given by

$$\pi(f \otimes g)(x, y) = f(x)g(y)$$

for $f, g \in R_{\mathbb{F}}(G)$ and $x, y \in G$. π is an $\mathbb{F}$-algebra morphism, and we show that it is an injection. Let $\alpha \in \ker(\pi)$, and write α as a finite sum $\alpha = \sum_j g_j \otimes h_j$. Let V be the $\mathbb{F}$-linear subspace of $R_{\mathbb{F}}(G)$ that is spanned by the elements g_j, and choose a basis $f_1, \cdots, f_m$ of V and elements $x_1, \cdots, x_m$ so that $f_j(x_i) = \delta_{ij}$ (Lemma 2.7). Then α can be rewritten as

$$\alpha = \sum_{j=1}^{m} f_j \otimes p_j.$$

For any $y \in G$ and $1 \le i \le m$, we have

$$\begin{aligned} 0 &= \pi(\alpha)(x_i, y) \\ &= \sum_{j=1}^{m} \pi(f_j \otimes p_j)(x_i, y) \\ &= p_i(y), \end{aligned}$$

and this shows that each $p_i = 0$. Consequently, $\alpha = 0$, proving that π is an injection. The multiplication on G induces an $\mathbb{F}$-algebra morphism:

$$\delta : R_{\mathbb{F}}(G) \to \mathbb{F}^{G\times G},$$

defined by

$$\delta(f)(x, y) = f(xy)$$

for $x, y \in G$. We have: $Im(\delta) \subset Im(\pi)$. In fact, let $f \in R_{\mathbb{F}}(G)$, and choose a basis $f_1, \cdots, f_n$ for the subspace of $R_{\mathbb{F}}(G)$ spanned by $G \cdot f$, and elements $x_1, \cdots, x_n \in G$, so that $f_i(x_j) = \delta_{ij}$. For $y \in G$, we have

$$y \cdot f = \sum_{i=1}^{n} g_i(y) f_i$$

where the g_i are some $\mathbb{F}$-valued functions on G. Evaluating the above equation at x_j, we obtain $g_j = f \cdot x_j$. Since $R_{\mathbb{F}}(G)$ is right-stable, we have $g_j \in R_{\mathbb{F}}(G)$. Then $\delta(f) = \pi(\sum_{i=1}^{n} f_i \otimes g_i) \in Im(\pi)$. Now the map

$$\gamma = \pi^{-1} \circ \delta : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(G) \otimes R_{\mathbb{F}}(G) \qquad (2.2.1)$$

is well defined, and it is an $\mathbb{F}$-algebra homomorphism. The relation between γ and the group multiplication of G may be seen by the equation

$$f(xy) = \sum_{i=1}^{n} f_i(x)g_i(y)$$

where

$$\gamma(f) = \sum_{i=1}^{n} f_i \otimes g_i$$

for $x, y \in G$. The $\mathbb{F}$-algebra $R_{\mathbb{F}}(G)$ becomes an $\mathbb{F}$-bialgebra with comultiplication γ and counit the evaluation map $c : f \mapsto f(1)$. We shall see shortly that $R_{\mathbb{F}}(G)$ is, in fact, a Hopf algebra.

Coefficient Functions Let $\rho : G \to GL(V, \mathbb{F})$ be a representation of G. For any linear function $\lambda \in V^*$ and $v \in V$, the function $\rho_{\lambda,v} : G \to \mathbb{F}$ defined by

$$\rho_{\lambda,v}(x) = \lambda(\rho(x)(v)), \quad x \in G,$$

is called a *coefficient function* of ρ. Let $[\rho]$ denote the subspace of the function space $\mathbb{F}^G$ that is spanned by all coefficient functions of ρ. From the identities:

$$x \cdot \rho_{\lambda,v} = \rho_{\lambda,\rho(x)(v)}; \quad \rho_{\lambda,v} \cdot x = \rho_{\lambda\circ\rho(x),v},$$

we see immediately that $[\rho]$ is bistable. The map

$$(\lambda, v) \mapsto \rho_{\lambda,v} : V^* \times V \to \mathbb{F}^G$$

is $\mathbb{F}$-bilinear, and since V is finite-dimensional, it follows that $[\rho]$ is also finite-dimensional. It follows from the observation above that all coefficient functions of ρ are representative functions of G, i.e. $[\rho] \subset R_{\mathbb{F}}(G)$.

Lemma 2.8 *For a group G, let V be a left stable finite-dimensional subspace of $R_{\mathbb{F}}(G)$, and let $\phi : G \to GL(V, \mathbb{F})$ be the representation of G by left translations on V. Then $[\phi]$ is exactly the subspace of $R_{\mathbb{F}}(G)$ spanned by the right translates of the elements of V. In particular, if V is bistable, then $V = [\phi]$.*

Proof. For $x \in G$, let x' denote the $\mathbb{F}$-linear function $f \mapsto f(x)$ on V. Then each $f \in V$ may be written as $f = \phi_{1',f} \in [\phi]$, so that we get $V \subset [\phi]$. Since $[\phi]$ is bistable, $[\phi]$ contains the subspace spanned by the right translates $V \cdot G$. To show that $[\phi]$ is spanned by $V \cdot G$, choose a basis $h_1, \cdots, h_n$ of V, and elements $x_1, \cdots, x_n$ of G so that the elements x_i' form the dual basis to $h_1, \cdots, h_n$ (Lemma 2.7). Then each $\phi_{x_i',h_j} = h_j \cdot x_i \in V \cdot G$, and since the elements ϕ_{x_i',h_j} span $[\phi]$, we see that $[\phi]$ is contained in the span of the right translates $V \cdot G$. ■

If $f \in R_{\mathbb{F}}(G)$, then by definition the left translates $G \cdot f$ span a finite-dimensional subspace, which is left stable. In view of Lemma 2.8, we then have

Corollary 2.9 $R_{\mathbb{F}}(G) = \bigcup_{\phi}[\phi]$, *where* ϕ *runs over all representations of* G. ■

Hopf Algebra $R_{\mathbb{F}}(G)$ For any function $f : G \to \mathbb{F}$, define the function $f' : G \to \mathbb{F}$ by $f'(x) = f(x^{-1}),\ x \in G$. If $\rho : G \to GL(V, \mathbb{F})$ is a representation, it is easy to verify that $f \mapsto f'$ maps $[\rho]$ ($\mathbb{F}$-linear) isomorphically onto $[\rho^0]$, where ρ^0 is the dual representation of ρ. Thus by Corollary 2.9, $R_{\mathbb{F}}(G)$ is stable under the map $f \mapsto f'$. Define

$$\eta : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(G)$$

by $\eta(f) = f'$. Then η is an involution, and we can easily see that η is the antipode of the $\mathbb{F}$-bialgebra $R_{\mathbb{F}}(G)$, i.e., $R_{\mathbb{F}}(G)$ is a Hopf algebra over $\mathbb{F}$.

A subset T of the $\mathbb{F}$-algebra $R_{\mathbb{F}}(G)$ is said to be *left* (resp. *right*) *stable* if $x \cdot T \subset T$ (resp. $T \cdot x \subset T$) for all $x \in G$. T is called *bistable* if it is both left and right stable. Finally, T is called *fully stable* if it is bistable and also stable under the involution η.

We have the following proposition, the proof of which follows easily from the definition.

Proposition 2.10 *A sub* $\mathbb{F}$*-algebra of* $R_{\mathbb{F}}(G)$ *is a sub bialgebra (resp. sub Hopf algebra) of* $R_{\mathbb{F}}(G)$ *if and only if it is bistable (resp. fully stable).* ■

2.3 Proper Automorphism Groups

Proper Automorphisms For a bistable subalgebra S of $R_{\mathbb{F}}(G)$, an $\mathbb{F}$-linear endomorphism of S is called *proper* if it commutes with the right translations $f \mapsto f \cdot x : S \to S$ for all $x \in G$. Let $End_{G-\mathbb{F}}(S)$ denote the set of all proper $\mathbb{F}$-linear endomorphisms of S. Then $End_{G-\mathbb{F}}(S)$ is a subalgebra of the $\mathbb{F}$-algebra $End_{\mathbb{F}}(S)$ of all $\mathbb{F}$-linear endomorphisms of S. Let $Aut_G(S)$ denote the group of all proper $\mathbb{F}$-algebra automorphisms of S. $Aut_G(S)$ is the group of units of the $\mathbb{F}$-algebra $End_{G-\mathbb{F}}(S)$. The left translation $\tau_y : f \mapsto y \cdot f$ by a (fixed) $y \in G$ is a proper $\mathbb{F}$-algebra automorphism of S, and hence $y \mapsto \tau_y$ defines a homomorphism $\tau : G \to Aut_G(S)$, called the *canonical homomorphism*.

Lemma 2.11 *Let S be a bistable subalgebra of $R_{\mathbb{F}}(G)$. Then any left stable $\mathbb{F}$-linear subspace of S is also α-stable for all $\alpha \in End_{G-\mathbb{F}}(S)$.*

Proof. Let V be a left stable linear subspace of S, and let $f \in V$. Choose a basis $f_1, \cdots, f_n$ for the subspace spanned by $G \cdot f$ and elements $x_1, \cdots, x_n$ of G such that $f_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq n$ (Lemma 2.7). For $x \in G$, $x \cdot f = \sum_{i=1}^{n} g_i(x) f_i$, where the g_i are $\mathbb{F}$-valued functions on G. Evaluating the above at x_j, we obtain $g_j = f \cdot x_j$, and hence we have

$$x \cdot f = \sum_{i=1}^{n} (f \cdot x_i)(x) f_i$$

$$f \cdot x = \sum_{i=1}^{n} f_i(x)(f \cdot x_i).$$

Applying α, we have

$$\begin{aligned} \alpha(f) \cdot x &= \alpha(f \cdot x) \\ &= \sum_{i=1}^{n} f_i(x) \alpha(f \cdot x_i) \\ &= \sum_{i=1}^{n} f_i(x) \alpha(f) \cdot x_i. \end{aligned}$$

Evaluating this at 1, we get

$$\alpha(f)(x) = (\alpha(f) \cdot x)(1)$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \alpha(f \cdot x_i)(1) f_i(x) \\
&= \sum_{i=1}^{n} \alpha(f)(x_i) f_i(x).
\end{aligned}
$$

Thus we have

$$\alpha(f) = \sum_{i=1}^{n} \alpha(f)(x_i) f_i \tag{2.3.1}$$

and $\alpha(f) \in V$, proving our lemma. ∎

A bistable subalgebra S of $R_{\mathbb{F}}(G)$ is an $\mathbb{F}$-bialgebra (Proposition 2.10), and $Hom_{\mathbb{F}}(S, \mathbb{F})$ becomes an $\mathbb{F}$-algebra under the convolution (§B.1). Define

$$\omega : End_{G-\mathbb{F}}(S) \to Hom_{\mathbb{F}}(S, \mathbb{F})$$

by $\omega(\alpha)(f) = \alpha(f)(1)$ for $\alpha \in End_{G-\mathbb{F}}(S)$ and $f \in S$. This is an $\mathbb{F}$-algebra isomorphism with its inverse

$$\zeta : Hom_{\mathbb{F}}(S, \mathbb{F}) \to End_{G-\mathbb{F}}(S)$$

given by

$$\zeta(\varphi)(f)(x) = \varphi(f \cdot x), \ \ \varphi \in Hom_{\mathbb{F}}(S, \mathbb{F}), \ \ f \in S, \ \ x \in G.$$

In fact, the verification of this statement is quite straightforward. For example, to see that ω is multiplicative, let $\alpha, \beta \in End_{G-\mathbb{F}}(S)$, and $f \in S$, and write

$$\gamma(f) = \sum_{i=1}^{n} g_i \otimes h_i,$$

where

$$\gamma : S \to S \otimes S$$

denotes the comultiplication of the bialgebra S. Then $f \cdot x = \sum_{i=1}^{n} g_i(x) h_i$. Applying the proper endomorphism β to this expression, and then evaluating at 1, we have

$$\beta(f)(x) = \sum_{i=1}^{n} \beta(h_i)(1) g_i(x),$$

and hence

$$\beta(f) = \sum_{i=1}^{n} \beta(h_i)(1)g_i.$$

Now, apply α to the above to get $\alpha(\beta(f)) = \sum_{i=1}^{n} \beta(h_i)(1)\alpha(g_i)$. Thus

$$\begin{aligned}
\omega(\alpha \circ \beta)(f) &= \alpha(\beta(f))(1) \\
&= \sum_{i=1}^{n} \alpha(g_i)(1)\beta(h_i)(1) \\
&= \sum_{i=1}^{n} \omega(\alpha)(g_i)\omega(\beta)(h_i) \\
&= (\omega(\alpha) * \omega(\beta))(f),
\end{aligned}$$

proving that $\omega(\alpha \circ \beta) = \omega(\alpha) * \omega(\beta)$.

Suppose now that the subalgebra S is fully stable. S is then a sub Hopf algebra of $R_{\mathbb{F}}(G)$ (Proposition 2.10), and $Hom_{\mathbb{F}-alg}(S, \mathbb{F})$ is a subgroup of the group of units of $Hom_{\mathbb{F}}(S, \mathbb{F})$ (see §B.1). The $\mathbb{F}$-algebra isomorphism ω maps the group $Aut_G(S)$ onto $Hom_{\mathbb{F}-alg}(S, \mathbb{F})$. We summarize what we have discussed above in the following proposition.

Proposition 2.12 *Suppose S is a fully stable subalgebra of $R_{\mathbb{F}}(G)$. The isomorphism ω above induces an isomorphism*

$$Aut_G(S) \cong Hom_{\mathbb{F}-alg}(S, \mathbb{F})$$

of groups. ∎

A bistable subalgebra S of $R_{\mathbb{F}}(G)$ is contained in the smallest fully stable subalgebra $\widehat{S}$ that contains S, namely, the intersection of all fully stable subalgebras of $R_{\mathbb{F}}(G)$ contain S. Below we shall show that there is a natural isomorphism

$$Aut_G(\widehat{S}) \cong Aut_G(S).$$

For that purpose, we first describe how to construct $\widehat{S}$ explicitly from S. Let V be any finite-dimensional bistable subspace of S, and let $\rho : G \to GL(V, \mathbb{F})$ be the representation of G on V by left translations. Let $d_V = \det_V \circ \rho$, where $\det_V : GL(V, \mathbb{F}) \to \mathbb{F}$ denotes the determinant function. Then d_V is a polynomial in elements of V. In fact, choose a basis $f_1, \cdots, f_m$ of V and elements $x_1, \cdots, x_m$ of G so that $f_j(x_i) = \delta_{i,j}$ (Lemma 2.7). Then the (i, j)-entry of the matrix of $\rho(x)$ with respect to the basis $f_1, \cdots, f_m$ is $(f_j \cdot x_i)(x)$. Thus each $f_j \cdot x_i \in V$, and $d_V \in S$ as d_V is a polynomial in the elements $f_j \cdot x_i$. We also note that $\frac{1}{d}$ is defined everywhere on G, and $\frac{1}{d} = d' \in R_{\mathbb{F}}(G)$. Let D denote the set of all d_V, where V runs over all finite-dimensional bistable subspaces of S. Then $\widehat{S}$ is the subalgebra of $R_{\mathbb{F}}(G)$ generated by the elements of S and the reciprocals of the elements of D. Note that if S is finitely generated as an $\mathbb{F}$-algebra, then so is $\widehat{S}$. Also note that S is $Aut_G(\widehat{S})$-stable by Lemma 2.11.

Proposition 2.13 *The restriction morphism*

$$Aut_G(\widehat{S}) \rightarrow Aut_G(S)$$

is an isomorphism of groups.

Proof. Clearly the map is an injection, and so we shall show that every proper $\mathbb{F}$-algebra automorphism of S extends to a proper $\mathbb{F}$-algebra automorphism of $\widehat{S}$. Let $\alpha \in Aut_G(S)$ and let $\phi \in Hom_{\mathbb{F}}(S, \mathbb{F})$ be the element corresponding to α. ϕ is given by $\phi(f) = \alpha(f)(1)$ for $f \in S$. Because of the isomorphism (Proposition 2.12)

$$Aut_G(\widehat{S}) \cong Hom_{\mathbb{F}-alg}(\widehat{S}, \mathbb{F})$$

it is enough to extend ϕ to an $\mathbb{F}$-algebra homomorphism $\widehat{S} \rightarrow \mathbb{F}$. Retaining the notation introduced in the construction of $\widehat{S}$ above, we first note $\phi(d) \neq 0$ for all $d \in D$. In fact, $\alpha(d) \neq 0$, so that $\alpha(d)(x) \neq 0$ for some $x \in G$. On the other hand, d is a homomorphism from G into $\mathbb{F}$, and hence $d \cdot x = d(x)d$. Thus

$$\begin{aligned} 0 \neq \alpha(d)(x) &= (\alpha(d) \cdot x)(1) \\ &= \alpha(d \cdot x)(1) = \alpha(d(x)d)(1) \\ &= d(x)\alpha(d)(1) = d(x)\phi(d), \end{aligned}$$

proving that $\phi(d) \neq 0$. Now let M be the (multiplicative) submonoid of S generated by D. Then $\widehat{S} = M^{-1}S$, and ϕ does not vanish on M. Hence ϕ extends to an $\mathbb{F}$-algebra homomorphism of $\widehat{S}$. ■

Pro-affine Algebraic Group Structure on $\mathbf{Aut_G(S)}$ Let G be a group, and for a fully stable subalgebra S of $R_{\mathbb{F}}(G)$, the group

$$\mathcal{G}(S) = Hom_{\mathbb{F}-alg}(S, \mathbb{F})$$

is given the structure of a pro-affine algebraic group with S as its polynomial algebra (Proposition B.1). Transferring the structure of a pro-affine algebraic group on $\mathcal{G}(S)$ to $Aut_G(S)$ by means of the natural isomorphism $\mathcal{G}(S) \cong Aut_G(S)$ (Proposition 2.12), $Aut_G(S)$ becomes a pro-affine algebraic group. In this case S is the algebra of polynomial functions of $Aut_G(S)$, where each $f \in S$ is viewed as a function on $Aut_G(S)$ by

$$f(\alpha) = \alpha(f)(1), \ \alpha \in Aut_G(S). \tag{2.3.2}$$

We often find it more advantageous to work with the group $Aut_G(S)$ rather than with $\mathcal{G}(S)$, because the group multiplication of $Aut_G(S)$ is the usual composition. If S is finitely generated, then $Aut_G(S)$ has the usual structure of an affine algebraic group.

Let $\tau : G \to Aut_G(S)$ be the canonical homomorphism defined at the beginning of the section. Since 0 (the zero function) is the only element of S vanishing on $\tau(G)$, we obtain the following.

Proposition 2.14 *$\tau(G)$ is Zariski dense in the pro-affine algebraic group $Aut_G(S)$.* ■

2.4 Analytic Representative Functions

Throughout this and the next section (i.e., §2.4-§2.5), we need to establish and discuss certain results that are true for both real and complex Lie groups. For that reason, the following convention will be in force in these two sections. A real or complex Lie group is simply referred to as a *Lie group*. For a Lie group G, the *analyticity* of a representation of G on a $\mathbb{C}$-linear space, or that of a $\mathbb{C}$-valued function of G, always refers to a real or complex one depending on whether G is a real or complex Lie group.

The Algebra of Analytic Representative Functions For a Lie group G, the set $R(G)$ of all $\mathbb{C}$-valued *analytic* representative functions on G forms a $\mathbb{C}$-linear subspace of the algebra $R_{\mathbb{C}}(G)$ of all

$\mathbb{C}$-valued representative functions on G. We denote this subspace by $R(G)$. Thus if G is a real analytic (resp. complex analytic) group, then $R(G)$ denotes the $\mathbb{C}$-linear space consisting of all $\mathbb{C}$-valued real analytic (resp. complex analytic) representative functions. For an analytic representation $\rho : G \to GL(V, \mathbb{C})$, the $\mathbb{C}$-linear span $[\rho]$ of all coefficient functions of ρ is a finite-dimensional bistable subspace of $R(G)$.

Proposition 2.15 *If G is a Lie group, then $R(G) = \bigcup_\rho [\rho]$, where ρ runs over all analytic representations of G, and $R(G)$ is a fully stable subalgebra of $R_{\mathbb{C}}(G)$.*

Proof. We have already seen above $\bigcup_\rho [\rho] \subset R(G)$. Now we show: $\bigcup_\rho [\rho] \supset R(G)$. Let $f \in R(G)$, and let V be the finite-dimensional linear space spanned by the left translations $G \cdot f$. The representation $\phi : G \to GL(V, \mathbb{C})$, given by $\phi(x)(g) = x \cdot g$, $x \in G$, $g \in V$, is analytic. In fact, choose a basis $h_1, \cdots, h_m$ of V and elements $x_1, \ldots, x_m$ of G such that $h_i(x_j) = \delta_{ij}$. For $x \in G$ and $g \in V$, we write

$$x \cdot g = \sum_i^m g_i(x) h_i$$

where each g_i is a $\mathbb{C}$-valued function on G. Applying the above to x_j, we obtain $g_j = g \cdot x_j$. Since g is an analytic function on G, so are its translations $g_j = g \cdot x_j$, and it follows that the representation ϕ is analytic. Since $V \subset [\phi]$ (Lemma 2.8), we have $f \in [\phi] \subset \bigcup_\rho [\rho]$.

To show that $R(G)$ is a $\mathbb{C}$-algebra, we merely note that if f and g are representative functions associated with analytic representations ρ and ϕ, respectively, then fg is a representative function associated with the tensor product $\rho \otimes \phi$. Since each $[\rho]$ is bistable, it follows that $R(G)$ is bistable. It now remains to show that $R(G)$ is fully stable. For this, it is enough to show that if ρ is an analytic representation of G, then the antipodal map $\eta : R_{\mathbb{C}}(G) \to R_{\mathbb{C}}(G)$ maps $[\rho]$ into $[\rho^\circ]$, where ρ° denotes the representation of G dual to ρ, which is necessarily analytic. Let $\lambda \in V^*$ and $v \in V$. Then $\eta(\rho_{\lambda,v}) = \rho^\circ_{v',\lambda}$, where v' denotes the linear functional on V^* given by $\mu \mapsto \mu(v)$. Since $R(G)$ is the union of all such $[\rho]$, $R(G)$ is stable under η, proving that $R(G)$ is fully stable. ∎

Proposition 2.16 *If G is a complex analytic group, then $R(G)$ is an integral domain.*

Proof. Suppose $f, g \in R(G)$ with $fg = 0$, and let $x_0 \in G$. Since f and g are analytic functions on G, either f or g is identically zero in a neighborhood of x_0 because the ring of formal power series is an integral domain. But if f is identically zero in some neighborhood of x_0, then f is zero in the whole of G by the principle of analytic continuation. ■

The following lemma shows that any analytic G-module can be constructed from $R(G)$.

Lemma 2.17 *Let ρ be an analytic representation of a Lie group G on a $\mathbb{C}$-linear space V. Then the G-module V is embedded as a sub G-module of the direct sum $[\rho] \oplus \cdots \oplus [\rho]$ ($\dim V$ copies), where $[\rho]$ is viewed as a left G-module, where G acts on $[\rho]$ by left translations.*

Proof. Let V^* denote the dual space of V, and for $\lambda \in V^*$ and $v \in V$, recall that the coefficient function $\rho_{\lambda,v} : G \to \mathbb{C}$ is defined by $\rho_{\lambda,v}(x) = \lambda(\rho(x)(v)),\ x \in G$. Choose a basis $\lambda_1, \lambda_2, \cdots, \lambda_n$ for the dual space V^*, and define $\phi : V \to [\rho] \oplus \cdots \oplus [\rho]$ by

$$\phi(v) = (\rho_{\lambda_1,v}, \cdots, \rho_{\lambda_n,v})$$

for $v \in V$. ϕ is clearly $\mathbb{C}$-linear and injective. Since $\rho_{\lambda_i,x\cdot v} = x \cdot \rho_{\lambda_i,v}$ for $x \in G$ $(1 \leq i \leq n)$, we see that ϕ is a morphism of G-modules. ■

Next we examine the relationship between the representative functions of a real analytic group G and those of G^+.

Proposition 2.18 *For any real analytic group G, the map $f \mapsto f \circ \gamma$ defines a $\mathbb{C}$-algebra isomorphism*

$$\alpha : R(G^+) \cong R(G),$$

where $\gamma : G \to G^+$ is the canonical map.

Proof. Clearly α is a morphism of $\mathbb{C}$-algebras. A real analytic representation ρ of G induces a complex analytic representation ρ^+ of G^+ with $\rho^+ \circ \gamma = \rho$, and any complex analytic representation of G^+ is obtained in this way. We first show that α is a $\mathbb{C}$-linear isomorphism. In light of Proposition 2.15, it is enough to show

$$f \mapsto f \circ \gamma : [\rho^+] \to [\rho]$$

is a $\mathbb{C}$-linear isomorphism, where $\rho : G \to GL(V, \mathbb{C})$ is any real analytic representation of G. For a $\mathbb{C}$-linear function λ of V and an element $v \in V$, the coefficient function $\rho_{\lambda,v}$ of ρ is the image of the coefficient function $\rho^{+}_{\lambda,v}$ of ρ^{+} under α, proving that $\alpha : [\rho^{+}] \to [\rho]$ is surjective. Now we prove that α is injective. Suppose $f \in [\rho^{+}]$ with $f \circ \gamma = 0$. For any $X \in \mathcal{L}(G)$, define $\phi : \mathbb{C} \to \mathbb{C}$ by

$$\phi(z) = f \circ \exp_{G^{+}}(d\gamma(X)). \tag{2.4.1}$$

Clearly ϕ is complex analytic, and we have $\phi(t) = 0$ for all $t \in \mathbb{R}$. Indeed, for $t \in \mathbb{R}$, we have

$$\phi(t) = f \circ \exp_{G^{+}}(d\gamma(tX)) = f \circ \gamma(exp_G(tX)) = 0.$$

Since the set of zeros of a nonzero complex analytic function on a connected open set in $\mathbb{C}$ is finite, $\phi(z) = 0$ for all $z \in \mathbb{C}$. On the other hand, $Im(d\gamma)$ spans $\mathcal{L}(G^{+})$, and hence the equation (2.4.1) and $\phi = 0$ imply that $f = 0$ in an open neighborhood of 1 in G^{+}. By the principle of analytic continuation, $f = 0$ on the entire G^{+}, proving that α is an injection. ∎

Example 2.19 We determine the algebra $R(\mathbb{C})$ of the complex group $\mathbb{C}$. The elements of $Hom(\mathbb{C}, \mathbb{C}^*)$ may be viewed as representations $\mathbb{C} \to GL(1, \mathbb{C})$, and the map

$$c \mapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : \mathbb{C} \to GL(2, \mathbb{C})$$

is a unipotent representation of $\mathbb{C}$. Hence $R(\mathbb{C})$ contains $Hom(\mathbb{C}, \mathbb{C}^*)$ and the identity map $z : \mathbb{C} \to \mathbb{C}$. We claim that $R(\mathbb{C})$ is generated (as a $\mathbb{C}$-algebra) by $Hom(\mathbb{C}, \mathbb{C}^*)$ and z, i.e.,

$$R(\mathbb{C}) \cong \mathbb{C}[Hom(\mathbb{C}, \mathbb{C}^*)] \otimes \mathbb{C}[z].$$

Let $\rho : \mathbb{C} \to GL(n, \mathbb{C})$ be a complex analytic representation of G, and identify the group $\mathbb{C}$ with its Lie algebra $\mathcal{L}(\mathbb{C})$. Then we have $\rho = \exp \circ\, d\rho$, where $d\rho : \mathbb{C} \to gl(n, \mathbb{C})$ denotes the differential of ρ. Let $d\rho(1) = S + N$ be the Jordan canonical decomposition of the matrix $d\rho(1)$, where S is a semisimple matrix, N a nilpotent matrix, and $SN = NS$. Thus, for $c \in \mathbb{C}$, we have $\rho(c) = \exp(cd\rho(1)) = \exp(cS)\exp(cN)$. Let $\varphi(c) = \exp(cS)$; $\psi(c) = \exp(cN)$. We first determine the representative functions associated with φ. Choose a matrix $A \in GL(n, \mathbb{C})$ such that $ASA^{-1} = diag(a_1, \cdots, a_n)$. Then

$$\begin{aligned} \exp(cS) &= \exp(A^{-1} diag(ca_1, \cdots, ca_n)A) \\ &= A^{-1}(\exp(diag(ca_1, \cdots, a_n)))A \\ &= A^{-1} diag(e^{ca_1}, \cdots, e^{ca_n})A. \end{aligned}$$

Thus the representative functions associated with φ are polynomials in the coefficient functions $c \mapsto e^{ca_i}$ $(1 \leq i \leq n)$, and hence we have $[\varphi] \subset \mathbb{C}[Hom(G, G^*)]$.

Next turning to the unipotent representation ψ, we have

$$\psi(c) = \exp(cN) = \sum_{i=0}^{\infty} \frac{(cN)^i}{i!} = \sum_{i=0}^{n} \frac{(cN)^i}{i!},$$

and this shows that the representative functions associated with ψ are polynomials in z, i.e., $[\psi] \subset \mathbb{C}[Hom(\mathbb{C}, \mathbb{C})] = \mathbb{C}[z]$. This completes the proof of our assertion. ■

2.5 Universal Algebraic Hull

In this section we continue to use the convention adopted at the beginning of §2.4.

For an analytic group G, let S be a fully stable subalgebra of $R(G)$. As we have seen in the preceding section, there is a canonical group isomorphism (Proposition 2.12)

$$Aut_G(S) \cong Hom_{\mathbb{C}-alg}(S, \mathbb{C})$$

through which the group $Aut_G(S)$ acquires the structure of a pro-affine algebraic group over $\mathbb{C}$ so that S is the polynomial algebra of $Aut_G(S)$.

Let V be a finite-dimensional bistable subspace of $R(G)$, and let S be the fully stable subalgebra of $R(G)$ that is generated by V. By Lemma 2.11, the restriction map $\alpha \mapsto \alpha_V$ defines a representation of $Aut_G(S)$ on V:

$$\phi : Aut_G(S) \to GL(V, \mathbb{C}).$$

Recalling that each $f \in V$ is viewed as a function on $Aut_G(S)$ (see the formula (2.3.2) in §2.3), we have $\alpha \cdot f = \alpha(f)$, where $\alpha \cdot f$ is the left translation of the function $f : Aut_G(S) \to \mathbb{C}$ by $\alpha \in Aut_G(S)$. This means that ϕ is exactly the representation of $Aut_G(S)$ on V by left translations, and ϕ is therefore a rational representation by Proposition B.2. Let A_V denote the image of ϕ, and let G_V denote the image of the canonical homomorphism $\rho : G \to GL(V, \mathbb{C})$, which maps $x \in G$ to the left translation $f \mapsto x \cdot f$.

Proposition 2.20 *A_V coincides with the Zariski closure G_V^* of G_V in $GL(V, \mathbb{C})$.*

Proof. Let $\tau : G \to Aut_G(S)$ be the canonical map. We have $\rho = \phi \circ \tau$, and $\tau(G)$ is Zariski dense in $Aut_G(S)$ (Proposition 2.14). Hence ϕ maps the Zariski dense subset $\tau(G)$ to $\rho(G) = G_V$, which is Zariski dense in $\phi(A) = A_V$ due to the rationality of ϕ. ∎

Universal Algebraic Hull If G is a (real or complex) analytic group, and if S is a fully stable subalgebra of $R(G)$, then the group $Aut_G(S)$ is a pro-affine algebraic group for which S is the algebra of the polynomial functions (see §2.3). Now we consider the $\mathbb{C}$-algebra $R(G)$ itself, and the resulting group $Aut_G(R(G))$ is denoted by $A(G)$. Thus the group $A(G)$ acquires the structure of a connected pro-affine algebraic group over $\mathbb{C}$ via canonical group isomorphism (Proposition 2.12)

$$A(G) \cong Hom_{\mathbb{C}-alg}(R(G), \mathbb{C}) \tag{2.5.1}$$

for which $R(G)$ is the algebra of polynomial functions on $A(G)$, i.e., $R(G) = P(A(G))$. We call $A(G)$ the *universal algebraic hull* of G. Note that every $f \in R(G)$ is viewed as a function $f : A(G) \to \mathbb{C}$ given by $f(\alpha) = \alpha(f)(1)$, $\alpha \in A(G)$.

The term *universal* above is justified by the following proposition.

Proposition 2.21 *Let G be an analytic group. Then the canonical homomorphism $\tau : G \to A(G)$ satisfies the following.*

(i) *$\tau(G)$ is Zariski dense in $A(G)$.*

(ii) *For any analytic representation $\rho : G \to GL(V, \mathbb{C})$, there exists a unique rational representation*

$$\widetilde{\rho} : A(G) \to GL(V, \mathbb{C})$$

such that $\widetilde{\rho} \circ \tau = \rho$.

Proof. The assertion (i) follows from Proposition 2.14. To prove (ii), identify the analytic G-module V with a sub G-module of a finite direct sum $[\rho] \oplus \cdots \oplus [\rho]$ of the G-module $[\rho]$ (Lemma 2.17). By Lemma 2.11, the bistable subspace $[\rho]$ of $R(G)$ is $A(G)$-stable, and hence we obtain a natural representation $A(G) \to GL([\rho], \mathbb{C})$, which is rational by what we have noted in the beginning of this section. Consequently, the action of $A(G)$ on the direct sum $[\rho] \oplus \cdots \oplus [\rho]$ is rational, and since the image $\tau(G)$ is Zariski dense in $A(G)$ by (i), the G-stable subspace V of $[\rho] \oplus \cdots \oplus [\rho]$ is also $A(G)$-stable. Thus we obtain a rational representation $\widetilde{\rho} : A(G) \to GL(V, \mathbb{C})$, and clearly we have $\widetilde{\rho} \circ \tau = \rho$. The uniqueness of $\widetilde{\rho}$ follows from (i). ■

Let V be a finite-dimensional bistable subspace of $R(G)$. Then we have seen that V is $A(G)$-stable (Lemma 2.11), and hence this induces a representation $A(G) \to GL(V, \mathbb{C})$. Let $A(G)_V$ denote the image of this representation, and let G_V denote the image of the representation of G on V by left translations. Then we have

Corollary 2.22 *$A(G)_V$ is the Zariski closure of G_V in $GL(V, \mathbb{C})$.*

Proof. Let $\widehat{V}$ denote the fully stable subalgebra of $R(G)$ that is generated by V. Since $R(G)$ is the polynomial algebra of $A(G)$, the restriction map $A(G) \to Aut_G(\widehat{V})$ is surjective by Theorem B.4. On the other hand, the restriction of $Aut_G(\widehat{V})$ to V coincides with the Zariski closure of G_V in $GL(V, \mathbb{C})$ by Proposition 2.20. Thus we see that $A(G)_V$ is the Zariski closure of G_V. ■

Topology and Analytic Structure of $A(G)$ If G is a Lie group such that $R(G)$ is finitely generated as a $\mathbb{C}$-algebra, the group $A(G)$ may be topologized in such a way that it becomes a Lie group. To give such a topology on $A(G)$, let $f_1, \cdots, f_m$ be a finite set generating the $\mathbb{C}$-algebra $R(G)$, and define

$$\phi : A(G) \to \mathbb{C}^m$$

by

$$\phi(\alpha) = (\alpha(f_1)(1), \cdots, \alpha(f_m)(1)), \quad \alpha \in A(G).$$

The map ϕ is an injection, and we now topologize $A(G)$ so that the bijection

$$\phi : A(G) \to \mathrm{Im}(\phi) \subset \mathbb{C}^m$$

becomes a homeomorphism, where the topology on the set $\mathrm{Im}(\phi)$ is the natural one induced from the topology of $\mathbb{C}^m$. We need to prove that the topology on $A(G)$ does not depend on the choice of the generators $f_1, \cdots, f_m$. Suppose $g_1, \cdots, g_n$ is another set of generators for $R(G)$, and define $\psi : A(G) \to \mathbb{C}^n$ by

$$\psi(\alpha) = (\alpha(g_1)(1), \cdots, \alpha(g_n)(1)), \ \alpha \in A(G).$$

Since the g_j generate $R(G)$, each f_i can be expressed as a polynomial in the functions $g_1, \cdots, g_n$, and similarly each g_j can be expressed as a polynomial in $f_1, \cdots, f_m$. Let P_i, Q_j be polynomials over $\mathbb{C}$ so that

$$\begin{aligned} f_i &= P_i(g_1, \cdots, g_n), \\ g_j &= Q_j(f_1, \cdots, f_m). \end{aligned} \tag{2.5.2}$$

Applying $\alpha \in A(G)$ to the equations in (2.5.2), and evaluating at the identity element 1, we get

$$\begin{aligned} \alpha(f_i)(1) &= P_i(\alpha(g_1)(1), \cdots, \alpha(g_n)(1)), \\ \alpha(g_j)(1) &= Q_j(\alpha(f_1)(1), \cdots, \alpha(f_m)(1)). \end{aligned} \tag{2.5.3}$$

Thus, if we put $\phi(\alpha) = (z_1, \cdots, z_m)$ and $\psi(\alpha) = (w_1, \cdots, w_n)$, it follows from (2.5.3) that each w_j is a polynomial in $z_1, \cdots, z_m$, and each z_i is a polynomial in $w_1, \cdots, w_n$, and we see that the bijection $\psi \circ \phi^{-1} : \mathrm{Im}(\phi) \to \mathrm{Im}(\psi)$ is a homeomorphism. We note that the topology on $A(G)$ is the weakest topology making the functions

$$\alpha \mapsto \lambda(\alpha(f)) : A(G) \to \mathbb{C}$$

continuous for all linear forms λ on $R(G)$ and all $f \in R(G)$. Now it is clear this topology makes $A(G)$ into a topological group.

Theorem 2.23 *Let G be an analytic group, and let V be a finite-dimensional bistable subspace of $R(G)$ which generates the algebra $R(G)$. The restriction map*

$$\alpha \mapsto \alpha_V : A(G) \to GL(V, \mathbb{C})$$

is an isomorphism $A(G) \cong A(G)_V$ as affine algebraic groups and also as complex analytic groups. In particular, $A(G)$ is a complex analytic group if $R(G)$ is finitely generated.

Proof. Note that each $\alpha \in A(G)$ leaves V invariant (Lemma 2.11), and hence the map $\alpha \mapsto \alpha_V$ is well defined. If G is a *complex* analytic group, then $R(G)$ is an integral domain (Proposition 2.16), and for a *real* analytic group G, we have $R(G) \cong R(G^+)$ (Proposition 2.18) and hence $R(G)$ is also an integral domain. This shows that the pro-affine algebraic group $A(G)$ is *connected* for either a real or complex analytic group G. Since the finite-dimensional $\mathbb{C}$-linear subspace V generates $R(G)$ as a $\mathbb{C}$-algebra, $R(G)$ is finitely generated, and hence the pro-affine group $A(G)$ is actually affine. By Proposition B.2, the surjective map $\alpha \mapsto \alpha_V$ is rational, and it is an isomorphism of affine algebraic groups, because V generates $R(G)$.

Next we show that $A(G) \cong A(G)_V$ as complex analytic groups. Choose a basis $h_1, \cdots, h_m$ of V, and let $x_1, \cdots, x_m \in G$ such that $h_i(x_j) = \delta_{ij}$. The matrix representation of the elements of $GL(V, \mathbb{C})$ with respect to the basis $h_1, \cdots, h_m$ allows us to identify $GL(V, \mathbb{C})$ with $GL(m, \mathbb{C})$. Then, for $\alpha \in A(G)$, α_V is identified with the matrix $(\alpha(h_j)(x_i))_{i,j}$, and we have seen that the map

$$\alpha \mapsto \alpha_V = (\alpha(h_j)(x_i))_{i,j} : A(G) \cong A(G)_V \qquad (2.5.4)$$

is an isomorphism of affine algebraic groups. Since any algebraic subgroup of $GL(V, \mathbb{C})$ is a closed subgroup of the Lie group $GL(V, \mathbb{C})$ as it is the set of zeros of some polynomial functions of $GL(V, \mathbb{C})$, we see that the connected algebraic subgroup $A(G)_V$ of $GL(V, \mathbb{C})$ is a complex analytic group under the Euclidean topology. Let ϕ denote the representation of G on V by left translations. Since V is bistable, we have $V = [\phi]$ by Lemma 2.8, and hence the coefficient functions $\phi_{x'_i, h_j}$ span the linear space V. Consequently the functions $\phi_{x'_i, h_j}$ generate the $\mathbb{C}$-algebra $R(G)$. Since

$$\alpha(\phi_{x'_i, h_j})(1) = \alpha(h_j \cdot x_i)(1) = (\alpha(h_j) \cdot x_i)(1) = \alpha(h_j)(x_i),$$

the isomorphism (2.5.4) becomes an isomorphism of topological groups by the way the group $A(G)$ is topologized. Consequently,

$$A(G) \cong A(G)_V \subset \mathbb{C}^{m^2}$$

as complex analytic groups.

For the last assertion of the theorem, we simply note that if $R(G)$ is finitely generated, there is a finite-dimensional bistable subspace V of $R(G)$ which generates $R(G)$. ∎

2.6 Relative Algebras

Suppose $\rho : G \to GL(V, \mathbb{C})$ is a complex analytic representation of a complex Lie group G, and let E be a closed complex Lie subgroup of G. We say that the representation ρ is *E-unipotent* if $\rho(E)$ is a unipotent subgroup of $GL(V, \mathbb{C})$. For $f \in R(G)$, we say that E acts *unipotently* on f (or simply f is *E-unipotent*), if the representation of G by left translations on the space spanned by the left translates $G \cdot f$ is E-unipotent. Let $R(G, E)$ denote the subset of $R(G)$ consisting of all E-unipotent functions $f \in R(G)$.

Lemma 2.24 *Let E be a closed normal complex Lie subgroup of a complex analytic group G. Then $R(G, E) = \bigcup_\rho [\rho]$, where ρ runs over all E-unipotent complex analytic representations of G.*

Proof. Let $f \in R(G, E)$ and let V_f denote the span (over $\mathbb{C}$) of the left translates $G \cdot f$. If $\phi : G \to GL(V_f, \mathbb{C})$ is the representation of G by left translations on V_f, then ϕ is E-unipotent, and $V_f \subset [\phi]$ by Lemma 2.8. This shows that $R(G, E) \subset \bigcup_\rho [\rho]$. Now suppose $\rho : G \to GL(V, \mathbb{C})$ is an E-unipotent complex analytic representation, and we show $[\rho] \subset R(G, E)$. For this, it is enough to prove that if σ denotes the representation of G by left translations on $[\rho]$, then σ is E-unipotent. Since ρ is E-unipotent, $\rho(x) - 1 : V \to V$ is nilpotent for $x \in E$. For any coefficient function $\rho_{\lambda,v}$ of ρ, where $\lambda \in V^*$ and $v \in V$, we have

$$(\sigma(x) - 1)(\rho_{\lambda,v}) = \rho_{\lambda,(\rho(x)-1)v}$$

for any $x \in G$. Since $[\rho]$ is spanned by the coefficient functions of ρ, the above identity readily implies that, for $x \in E$, $(\sigma(x) - 1)$ acts nilpotently on $[\rho]$. ■

The tensor product of two E-unipotent representations are again E-unipotent. Moreover, if a complex analytic representation ρ of a complex Lie group G is E-unipotent, then so is its dual ρ°, and the antipodal map η maps $[\rho]$ into $[\rho^\circ]$. Lemma 2.24 thus ensures that $R(G, E)$ is a fully stable subalgebra of $R(G)$.

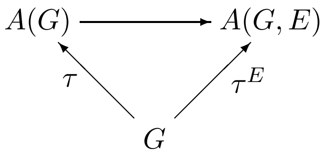
We define

$$A(G, E) = Aut_G(R(G, E)).$$

By Theorem B.4, the restriction map $A(G) \to A(G, E)$ is surjective, i.e., $A(G, E) = A(G)_{R(G,E)}$. $A(G, E)$ is a pro-affine algebraic group, and we have a canonical homomorphism

$$\tau^E : G \to A(G, E),$$

which is given by $\tau^E(x)(f) = x \cdot f$ for $x \in G$ and $f \in R(G, E)$. We note that τ^E maps G onto a Zariski dense subgroup of $A(G, E)$. This follows easily from the commutativity of the diagram



We have the following universal property of τ^E (cf., Proposition 2.21).

Proposition 2.25 *If $\rho : G \to GL(V, \mathbb{C})$ is an E-unipotent complex analytic representation, there is a unique rational representation*

$$\widetilde{\rho} : A(G, E) \to GL(V, \mathbb{C})$$

such that $\widetilde{\rho} \circ \tau^E = \rho$. ∎

Suppose $\phi : G \to H$ is a morphism of complex Lie groups, and let E and L be closed normal complex Lie subgroups of G and H, respectively, such that $\phi(E) \subset L$. The induced algebra morphism $f \mapsto f \circ \phi : R(H) \to R(G)$ maps $R(H, L)$ into $R(G, E)$, and hence defines a $\mathbb{C}$-algebra morphism $\phi^* : R(H, L) \to R(G, E)$. The map ϕ^*, in turn, induces a morphism of pro-affine algebraic groups,

$$\widehat{\phi} : Hom_{\mathbb{C}-alg}(R(G, E), \mathbb{C}) \to Hom_{\mathbb{C}-alg}(R(H, L), \mathbb{C}).$$

By means of the isomorphisms

$$Hom_{\mathbb{C}-alg}(R(G, E), \mathbb{C}) \cong A(G, E);$$
$$Hom_{\mathbb{C}-alg}(R(H, L), \mathbb{C}) \cong A(H, L),$$

we obtain a morphism

$$\widehat{\phi} : A(G, E) \to A(H, L)$$

of pro-affine algebraic groups. This is the morphism determined uniquely by the commutativity of the diagram

$$\begin{array}{ccc} G & \xrightarrow{\phi} & H \\ {\scriptstyle \tau_G}\downarrow & & \downarrow{\scriptstyle \tau_H} \\ A(G,E) & \xrightarrow[\widehat{\phi}]{} & A(H,L) \end{array}$$

where τ_G and τ_H are the canonical maps. For any $\alpha \in A(G,E)$, $\widehat{\phi}(\alpha)$ is related to α by the commutative diagram

$$\begin{array}{ccc} R(H,L) & \xrightarrow{\phi^*} & R(G,E) \\ {\scriptstyle \widehat{\phi}(\alpha)}\downarrow & & \downarrow{\scriptstyle \alpha} \\ R(H,L) & \xrightarrow[\phi^*]{} & R(G,E) \end{array}$$

Proposition 2.26 *Let $G' \xrightarrow{\theta} G \xrightarrow{\pi} G'' \to 1$ be an exact sequence of complex analytic groups, and let E' be a closed normal complex Lie subgroup of G' such that $E = \theta(E')$ is a closed normal complex Lie subgroup of G. Then the induced sequence*

$$A(G',E') \xrightarrow{\widehat{\theta}} A(G,E) \xrightarrow{\widehat{\pi}} A(G'') \to 1$$

is exact.

Proof. We first note that since $E = \theta(E')$, π induces an algebra morphism $\pi^* : R(G'') \to R(G,E)$, and hence $\widehat{\pi} : A(G,E) \to A(G'')$. Now we have the commutative diagram:

$$\begin{array}{ccccccc} G' & \xrightarrow{\theta} & G & \xrightarrow{\pi} & G'' & \longrightarrow & 1 \\ {\scriptstyle \tau'}\downarrow & & {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tau''} & & \\ A(G',E') & \xrightarrow[\widehat{\theta}]{} & A(G,E) & \xrightarrow[\widehat{\pi}]{} & A(G'') & \longrightarrow & 1 \end{array}$$

Since the algebraic subgroup $\widehat{\pi}(A(G,E))$ contains the Zariski dense subgroup $\tau''(G'') = \widehat{\pi}(\tau(G))$ of $A(G'')$, $\widehat{\pi}$ is surjective. To show $\mathrm{Im}(\widehat{\theta}) = \ker(\widehat{\pi})$, we first show that

$$\mathrm{Im}(\pi^*) = R(G,E)^{\theta(G')}.$$

Since $\pi \circ \theta = 1$, the inclusion $\mathrm{Im}(\pi^*) \subset R(G,E)^{\theta(G')}$ is clear. For the reverse inclusion, let $f \in R(G,E)^{\theta(G')}$. For $x' \in G'$ and $y \in G$, we have $f(y\theta(x')) = (\theta(x')f)(y) = f(y)$, and hence f is constant on each left coset of the subgroup $\theta(G') = \ker(\pi)$ in G. Thus there exists a unique $f'' : G'' \to \mathbb{C}$ so that $f'' \circ \pi = f$. It remains to show $f'' \in R(G'')$. The map

$$g \mapsto g \circ \pi : \mathbb{C}^{G''} \to \mathbb{C}^G$$

is an injective morphism of G-modules, where G acts on the function space $\mathbb{C}^{G''}$ by left translations via π and on $\mathbb{C}^G$ by left translations. Since $f \in R(G)$, the left translates $G \cdot f$ span a finite-dimensional $\mathbb{C}$-linear space, and hence $G \cdot f''$ spans a finite-dimensional subspace of $\mathbb{C}^{G''}$, proving $f'' \in R(G'')$.

Next we note that, for $f \in R(G,E)$ and $\alpha \in A(G,E)$, we always have $\alpha \cdot f = \alpha(f)$, where $\alpha \cdot f$ denotes the left translation of f by α. From the commutative diagram

$$\begin{CD} R(G'') @>{\pi^*}>> R(G,E) \\ @V{\widehat{\pi}(\alpha) = I}VV @VV{\alpha}V \\ R(G'') @>>{\pi^*}> R(G,E) \end{CD}$$

it follows that

$$\alpha \in \ker(\widehat{\pi}) \text{ if and only if } \alpha = I \text{ on } \mathrm{Im}(\pi^*) = R(G,E)^{\theta(G')},$$

and the latter happens if and only if

$$\alpha \cdot f = f \text{ for all } f \in R(G,E)^{\theta(G')}.$$

On the other hand, $\tau(\theta(G')) = \widehat{\theta}(\tau'(G'))$, and its Zariski closure in $A(G,E)$ is $\widehat{\theta}(A(G',E'))$, and this implies

$$R(G,E)^{\theta(G')} = R(G,E)^{\widehat{\theta}(A(G',E'))}.$$

Since the subgroup $\tau(\theta(G'))$ is normal in $\tau(G)$ and since $\tau(G)$ is Zariski dense in $A(G,E)$, the Zariski closure $\widehat{\theta}(A(G',E'))$ of $\tau(\theta(G'))$ is normal in $A(G,E)$. The fixer of $R(G,E)^{\widehat{\theta}(A(G',E'))} = R(G,E)^{\theta(G')}$ in the pro-affine algebraic group $A(G,E)$ coincides with the normal algebraic subgroup $\widehat{\theta}(A(G'),E')$ (see §B.2). Therefore $\alpha \in \ker(\widehat{\pi})$ if and only if $\alpha \in \widehat{\theta}(A(G',E'))$, proving that $\mathrm{Im}(\widehat{\theta}) = \ker(\widehat{\pi})$. ∎

2.7 Unipotent Hull

Given an analytic group G, the unipotent radical $U(G)$ of the pro-affine algebraic group $A(G)$ is called the *unipotent hull* of G. This is a normal unipotent subgroup of $A(G)$, which is maximal in the sense that it contains every normal unipotent subgroup of $A(G)$. $A(G)$ has a semidirect product decomposition $A(G) = U(G) \cdot M$, where M is a maximal reductive subgroup of $A(G)$ (Theorem B.8).

Semisimple Representative Functions For $f \in R(G)$, let V_f denote the subspace spanned over $\mathbb{C}$ by the left translates $G \cdot f$. It is clearly an S-stable finite-dimensional subspace of $R(G)$. f is called a *semisimple element* if the representation ρ_f of G by left translations on V_f is semisimple. Let $R(G)_s$ denote the subset consisting of all semisimple elements in $R(G)$. We claim

$$R(G)_s = \bigcup_{\rho} [\rho] \tag{2.7.1}$$

where ρ runs over all semisimple complex analytic representations of G. If $f \in R(G)_s$, then ρ_f is semisimple, and $f \in [\rho_f] \subset \bigcup_\rho [\rho]$, so that we have $R(G)_s \subset \bigcup_\rho [\rho]$. Suppose now that ρ is a semisimple analytic representation on V. To show $[\rho] \subset R(G)_s$, it is enough to show that the coefficient functions of ρ are semisimple. Let $\lambda \in V^*$ and $v \in V$, and consider the coefficient function $\rho_{\lambda,v}$. Let W be the sub G-module of the G-module V generated by v, and define $\phi : W \to R(G)$ by $\phi(w) = \rho_{\lambda,w}$, $w \in W$. Then W is a semisimple G-module, and the linear map ϕ commutes with the (left) actions of G on W and $R(G)$. Hence $\phi(W)$ is semisimple. On the other hand, the sub G-module of $R(G)$, which is spanned by the left translates $G \cdot \rho_{\lambda,v}$, is exactly the image of ϕ. Since $x \cdot \rho_{\lambda,v} = \rho_{\lambda,xv}$, $x \in G$, we see that $\rho_{\lambda,v}$ is semisimple, i.e., $\rho_{\lambda,v} \in R(G)_s$. Since the direct sum and the tensor product of any two semisimple representations of G are again semisimple (Theorem 2.1), the identity (2.7.1) above implies that $R(G)_s$ is a subalgebra of $R(G)$.

By Proposition B.9, we have

Proposition 2.27 *The unipotent hull $U(G)$ of an analytic group G is the intersection of the kernels of all restriction homomorphisms*

$$A(G) \to A(G)_{[\rho]},$$

where ρ runs over all semisimple analytic representations of G. ■

If G is reductive, every analytic representation of G is semisimple and hence $U(G) = (1)$ by Proposition 2.27. More generally, suppose that E is a closed normal complex Lie subgroup of G, and denote the unipotent radical of the pro-affine algebraic group $A(G, E)$ by $U(G, E)$. $U(G, E)$ is the intersection of all kernels of the restriction homomorphisms

$$A(G, E) \to A(G, E)_{[\rho]}$$

where ρ runs over all semisimple analytic representations of G that are trivial on E.

Example 2.28 We have (see Example 2.19)

$$R(\mathbb{C}) \cong \mathbb{C}[Hom(\mathbb{C}, \mathbb{C}^*)] \otimes \mathbb{C}[z],$$

and hence

$$\begin{aligned} U(\mathbb{C}) &= \ker(A(\mathbb{C}) \to A(\mathbb{C})_{\mathbb{C}[Hom(\mathbb{C},\mathbb{C}^*)]}) \\ &\cong Hom_{\mathbb{C}-alg}(\mathbb{C}[z], \mathbb{C}) \cong \mathbb{C}. \end{aligned}$$

■

Let E and L be closed normal complex Lie subgroups of complex Lie groups G and H, respectively. If $\phi : G \to H$ is a morphism of complex Lie groups such that $\phi(E) \subset L$, then the induced morphism $\widehat{\phi} : A(G, E) \to A(H, L)$ of pro-affine algebraic groups does not, in general, map $U(G, E)$ into $U(H, L)$. However, we have

Lemma 2.29 $\widehat{\phi}(U(G, E)) \subset U(H, L)$ *holds in each of the following cases:*

(i) $\phi(G)$ *is a normal subgroup of* H*;*

(ii) H *is solvable.*

Proof. Assume the condition (i), and let $\tau_G : G \to A(G, E)$ and $\tau_H : H \to A(H, L)$ denote the canonical maps. Since $\tau_G(G)$ is Zariski dense in $A(G, E)$, so is $\widehat{\phi}(\tau_G(G)) = \tau_H(\phi(G))$ in $\widehat{\phi}(A(G, E))$. On the other hand, the normalizer N of $\tau_H(\phi(G))$ in $A(H, L)$ is Zariski closed in $A(H, L)$ and contains $\tau_H(H)$ due to the normality of $\phi(G)$ in H. It follows from the density of $\tau_H(H)$ in $A(H,L)$ that $N = A(H,L)$, proving that $\widehat{\phi}(A(G,E))$ is normal in $A(H,L)$. As a unipotent normal subgroup of $A(G,E)$, the group $\widehat{\phi}(U(G,E))$ is contained in $U(H,L)$.

Suppose (ii) holds. For any bistable finite-dimensional subspace V of $R(H,L)$, the algebraic group $A(H,L)_V$ is the Zariski-closure H_V^* of the image H_V under the canonical map $H \to GL(V,\mathbb{C})$. Hence $A(H,L)_V$ is a solvable linear algebraic group, and thus its unipotent radical U_V consists of *all* the unipotent elements of $A(H,L)_V$. If W is another finite-dimensional bistable subspace of $R(H,L)$ with $W \subset V$, then the restriction morphism $A(H,L)_W \to A(H,L)_V$ maps the unipotent radical U_W onto the unipotent radical U_V. Then we have

$$A(H,L) = \lim_{\longleftarrow} A(H,L)_V \ ; \ U(H,L) = \lim_{\longleftarrow} U_V,$$

and hence $U(H,L)$ consists of all the unipotent elements in $A(H,L)$. Since the rational morphism $\widehat{\phi}$ sends unipotent elements to unipotent elements, $\widehat{\phi}(U(G)) \subset U(H)$ follows. ■

Corollary 2.30 *Let K be a closed normal complex analytic subgroup of a complex analytic group G. If E is a closed normal complex Lie subgroup of G such that $E \subset K$, the inclusion $i : K \subset G$ induces an exact sequence*

$$U(K,E) \xrightarrow{\widehat{i}} U(G,E) \to U(G/K) \to 1.$$

Proof. We have an exact sequence (see §2.5)

$$A(K,E) \xrightarrow{\widehat{i}} A(G,E) \xrightarrow{\widehat{\pi}} A(G/K) \to 1,$$

where $\widehat{\pi}$ is induced by the canonical map $\pi : G \to G/K$. By the lemma above, $\widehat{i}(U(K,E)) \subset U(G,E)$. On the other hand, $\widehat{\pi}$ maps $U(G,E)$ onto $U(G/K)$ by Proposition B.7. We clearly have $\widehat{i}(U(K,E)) \subset \ker(\widehat{\pi}|_{U(G,E)})$. To prove the reverse inclusion, let $\alpha \in U(G,E)$ with $\widehat{\pi}(\alpha) = 1$, and let $\beta \in A(K,E)$ such that $\widehat{i}(\beta) = \alpha$. Choose maximal reductive algebraic subgroups M' and M of $A(K,E)$ and $A(G,E)$, respectively, so that $\widehat{i}(M') \subset M$. Since we have $A(K,E) = U(K,E) \cdot M'$ (semidirect product), write $\beta = \upsilon\gamma$ with $\upsilon \in U(K,E)$ and $\gamma \in M'$. Then

$$\widehat{i}(\upsilon)^{-1}\alpha = \widehat{i}(\gamma) \in U(G,E) \cap M = (1),$$

proving that $\alpha = \widehat{i}(v) \in \widehat{i}(U(K,E))$. ■

In general, the inclusion $i : K \subset G$ does not induce an injection $\widehat{i} : U(K,E) \to U(G,E)$. The following lemma, however, provides a sufficient condition for $\widehat{i}$ to be injective.

Lemma 2.31 *Let K be a closed normal complex analytic subgroup of a complex analytic group G, and let E be a closed normal complex analytic subgroup of G with $E \subset K$. Suppose that S is a bistable subspace of $R(K,E)$ such that*

(i) *S separates the points of $U(K,E)$, and*

(ii) *$S \subset \mathrm{Im}(R(G,E) \xrightarrow{\iota^*} R(K,E))$.*

Then $\widehat{\iota} : U(K,E) \to U(G,E)$ is an injection.

Proof. Since S separates the points of $U(K,E)$, the restriction map $U(K,E) \to U(K,E)_S$ is an isomorphism. To show $\widehat{\iota}$ is an injection, let $\alpha \in U(K,E)$, $\widehat{\iota}(\alpha) = I$. From the commutativity of the diagram

$$\begin{array}{ccc} R(G,E) & \xrightarrow{\iota^*} & R(K,E) \\ \Big\downarrow I & & \Big\downarrow \alpha \\ R(G,E) & \xrightarrow[\iota^*]{} & R(K,E) \end{array}$$

we obtain $\iota^*(f) = \alpha(\iota^*(f))$ for all $f \in R(G,E)$. This means $\alpha = I$ on $\mathrm{Im}(\iota^*)$, and since $S \subset \mathrm{Im}(\iota^*)$, we have $\alpha_S = I$, and hence $\alpha = I$, proving that $\widehat{\iota}$ is injective. ■

Chapter 3

Extensions of Representations

In this chapter we shall deal with the problem of determining when an analytic representation of a subgroup of a complex analytic group is extendable to the entire group ([12], [25]).

Let L be a complex Lie subgroup of a complex Lie group G, and let ρ be a complex analytic representation of L on a $\mathbb{C}$-linear space V. A representation σ of G is said to be an *extension* of the representation ρ of L, if the representation space W of σ contains V as a G-stable subspace and $\sigma(x)$ coincides with $\rho(x)$ on V for all $x \in L$. If the representation ρ of L has such an extension, then we shall simply say that ρ is *extendable* to G.

3.1 Some Examples

We present two examples, each of which shows that the extension of a representation of a subgroup to the whole group is *not* possible.

(1) Let $G = \mathbb{C} \rtimes \mathbb{C}$ be the semidirect product of the additive group $\mathbb{C}$ by itself, where $\mathbb{C}$ acts on itself by $(t, z) \to e^t z : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. The multiplication in G is given by $(x, s)(y, t) = (x + e^s y, s + t)$. This is a complex analytic group isomorphic with the subgroup of $GL(3, \mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} e^t & 0 & z \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

where $t, z \in \mathbb{C}$. We identify G with this linear group. Then the Lie algebra $\mathfrak{g}$ of G is the subalgebra of $\mathfrak{gl}(3, \mathbb{C})$ consisting of all matrices of the form

$$\begin{pmatrix} t & 0 & x \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$$

where $x, t \in \mathbb{C}$.

Let H be the subgroup of G consisting of all elements

$$\begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{C}$. Then no nontrivial 1-dimensional representation of H is extendable to a representation of G. In fact, let $\rho : H \to GL(W, \mathbb{C})$ be a nontrivial 1-dimensional complex analytic representation of H, which extends to a complex analytic representation, σ say, of G on a finite-dimensional $\mathbb{C}$-linear space V. Choose a nonzero element $w \in W$ as a basis of W so that $W = \mathbb{C}w$ is a 1-dimensional $\sigma(H)$-stable subspace of V. The differential $d\sigma$ is a representation of the Lie algebra $\mathfrak{g}$ of G, and it extends the differential $d\rho$ of ρ.

Let

$$e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\{e, f\}$ is a basis of $\mathfrak{g}$, and $\{e\}$ is a basis for $\mathcal{L}(H)$. Since ρ is nontrivial, there is a nonzero $c \in \mathbb{C}$ so that

$$d\sigma(e)(w) = d\rho(e)(w) = cw.$$

On the other hand, $[f, e] = e \in [\mathfrak{g}, \mathfrak{g}]$, and since $\mathfrak{g}$ is solvable, $d\sigma([\mathfrak{g}, \mathfrak{g}])$ is nilpotent on V by Lie's Theorem (Theorem A.6). Thus $d\sigma(e)$ is nilpotent on V, and 0 is hence the only eigenvalue of $d\sigma(e)$. Therefore $c = 0$, a contradiction.

(2) Let $G = SL(2, \mathbb{C})$, and let H be the subgroup of G consisting of all elements

$$\begin{pmatrix} e^t & s \\ 0 & e^{-t} \end{pmatrix}$$

with $s, t \in \mathbb{C}$. Let V_0 be a 1-dimensional $\mathbb{C}$-linear space, and let $v_0 \in V_0$, $v_0 \neq 0$, so that $V_0 = \mathbb{C}v_0$. Fix a *positive* number λ, and define the representation

$$\rho : H \to GL(V_0, \mathbb{C})$$

by

$$\rho(\begin{pmatrix} e^t & s \\ 0 & e^{-t} \end{pmatrix})(v_0) = e^{-\lambda t} v_0.$$

Then ρ cannot be extended to an analytic representation of G. To see this, suppose ρ is extendable to an analytic representation of G. Then the differential

$$d\rho : \mathcal{L}(H) \to \mathfrak{gl}(V_0, \mathbb{C})$$

of ρ has an extension to a representation ϕ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ on a finite-dimensional $\mathbb{C}$-linear space V. Choose the basis $\{e, f, h\}$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of G, where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We have the bracket relations

$$[h, e] = 2e, \ [h, f] = -2f, \ [e, f] = h.$$

We also have $\phi(h)(v_0) = -\lambda v_0$ and $\phi(e)(v_0) = 0$. In fact, since $\rho = 1$ on the elements

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix},$$

for all $c \in \mathbb{C}$, we have $\phi(e)(v_0) = d\rho(e)(v_0) = 0$. For the first assertion,

$$\begin{aligned} \phi(h)(v_0) &= d\rho(h)(v_0) \\ &= \frac{d}{dt}\rho(\begin{pmatrix} e^t & s \\ 0 & e^{-t} \end{pmatrix})(v_0)|_{t=0} \\ &= \frac{d}{dt}e^{-\lambda t}v_0|_{t=0} \\ &= -\lambda v_0. \end{aligned}$$

For each integer $i \geq 0$, define

$$v_i = \phi(f)^i(v_0).$$

Then v_i is either 0 or an eigenvector of $\phi(h)$ with eigenvalue $-(\lambda+2i)$. To prove this, we use induction on i. Thus assume

$$\phi(h)(v_i) = -(\lambda + 2i)v_i.$$

Then

$$\begin{aligned}
\phi(h)(v_{i+1}) &= \phi(h) \circ \phi(f)(v_i) \\
&= \phi(f) \circ \phi(h)(v_i) + \phi([h, f])(v_i) \\
&= \phi(f)(-(\lambda + 2i)v_i) - 2\phi(f)(v_i) \\
&= -(\lambda + 2(i + 1))\phi(f)(v_i) \\
&= -(\lambda + 2(i + 1))v_{i+1},
\end{aligned}$$

completing the induction. Let m be the largest integer with $v_m \neq 0$, and let W be the subspace of V that is spanned by $v_0, v_1, \cdots, v_m$. Again using induction we may verify

$$\phi(e)(v_i) = -i(\lambda + (i - 1))v_{i-1},\ i = 1, \cdots, m,$$

and from this we deduce that W is stable under $\phi(e)$. Consequently, W is stable under $\phi(\mathfrak{sl}(2, \mathbb{C}))$. Now we consider the linear map $\phi(h)_W : W \to W$. Since

$$Tr(\phi(h)_W) = Tr(\phi(e)_W \circ \phi(f)_W - \phi(f)_W \circ \phi(e)_W) = 0$$

we have

$$\sum_{i=0}^{m} -(\lambda + 2i) = -(m + 1)(\lambda + m) = 0$$

Thus $\lambda + m = 0$ and $\lambda = -m \leq 0$. Since λ was chosen to be positive, we have a contradiction.

3.2 Decomposition Theorem

The main purpose of this section is to study how the decomposition of an analytic group as a semidirect product affects the decomposition of its algebra of representative functions.

Lemma 3.1 *Let H and N be closed complex analytic subgroups of a complex analytic group G with N normal in G, and let E be a closed* *normal complex Lie subgroup of* G *such that* $E \subset N$. *For* $h \in H$, *let* $\kappa(h)$ *denote the automorphism of* N *given by* $n \mapsto hnh^{-1}$. *Then for* $g \in R(N, E)$, $g \circ \kappa(H) \subset R(N, E)$.

Proof. For $h \in H$ and $n \in N$, we have

$$n \cdot (g \circ \kappa(h)) = (\kappa(h)(n) \cdot g) \circ \kappa(h) \tag{3.2.1}$$

or, equivalently,

$$(n \cdot g) \circ \kappa(h) = \kappa(h^{-1})(n)(g \circ \kappa(h)). \tag{3.2.2}$$

For a fixed h, and with n ranging over N, the equation (3.2.1) shows that $g \circ \kappa(h) \in R(N)$. To show $g \circ \kappa(h) \in R(N, E)$, let V and W be the finite-dimensional subspaces of $R(N)$ that are spanned by $N \cdot g$ and $N \cdot (g \circ \kappa(h))$, respectively, and let $\theta : V \to W$ be the map that sends f to $f \circ \kappa(h)$. Then θ is a $\mathbb{C}$-linear isomorphism by the formula (3.2.1). If ρ and ϕ are the representations of N by left translations on V and W, respectively, then we have

$$\phi(z) = \theta \circ \rho(hzh^{-1}) \circ \theta^{-1}$$

for $z \in N$, i.e., we have the commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{\rho(hzh^{-1})} & V \\ \theta \downarrow & & \downarrow \theta \\ W & \xrightarrow[\phi(z)]{} & W \end{array}$$

Hence, for any $z \in N$ and any positive integer m, we have

$$(\phi(z) - 1)^m = \theta \circ (\rho(hzh^{-1}) - 1)^m \circ \theta^{-1}.$$

Since $g \in R(N, E)$, $\rho(E)$ acts on V unipotently. $\phi(E)$ therefore acts unipotently on W, proving $g \circ \kappa(h) \in R(N, E)$. ■

Let G be a complex analytic group, and let H and N be closed complex analytic subgroups of G with N normal in G such that $G = H \cdot N$ is a semidirect product. For any function $f : N \to \mathbb{C}$ (resp. $f : H \to \mathbb{C}$), define the function $f^+ : G \to \mathbb{C}$ by $f^+(hn) = f(n)$ (resp.

$f^+(hn) = f(h))$ for $n \in N$ and $h \in H$. For $h \in H$, let $\kappa(h)$ denote the automorphism of N given by $n \mapsto hnh^{-1}$.

Let E be a closed normal complex Lie subgroup of G with $E \subset N$. Let $R(G,E)_N$ denote the image of $R(G,E)$ under the restriction map $R(G,E) \to R(N,E)$. Then we have

Lemma 3.2 *For $g \in R(N,E)$, the following are equivalent.*

(i) $g \in R(G,E)_N$.

(ii) $g \circ \kappa(H) = \{g \circ \kappa(h) : h \in H\}$ *spans a finite-dimensional subspace of* $R(N,E)$.

(iii) $g^+ \in R(G,E)$.

Proof. (iii)⇒(i) follows from $g^+|_N = g$.

(i)⇒(ii): Let $f \in R(G,E)$ with $f_N = g$. Then, for $h \in H$, we have

$$g \circ \kappa(h) = (h^{-1} \cdot f \cdot h)_N.$$

Now, f being a representative function of G, the set $\{h^{-1} \cdot f \cdot h\}$ spans a finite-dimensional space, and hence the subspace of $R(N,E)$ spanned by $g \circ \kappa(H)$ is finite-dimensional.

(ii)⇒(iii): Since $g \circ \kappa(H) \subset R(N,E)$ (Lemma 3.1), the $\mathbb{C}$-linear span V of the left translates $N \cdot (g \circ \kappa(H))$ is a finite-dimensional left-stable subspace of $R(N,E)$. Now we define $\rho : G \to GL(V,\mathbb{C})$ by

$$\rho(z)(f) = (n \cdot f) \circ \kappa(h^{-1}),$$

where $z \in G$, $z = hn$ with $n \in N$ and $h \in H$, and $f \in V$. By the formula (3.2.2), the above map is well defined, and we see that ρ is a complex analytic representation of G. Since $V \subset R(N,E)$, it is clear that ρ is E-unipotent. Let $\varepsilon : V \to \mathbb{C}$ be the linear function given by $\varepsilon(f) = f(1)$ for all $f \in V$. Then the extension function g^+ of g is exactly the coefficient function $\rho_{\varepsilon,g}$, which is, by definition, given by $x \mapsto \varepsilon(\rho(x)(g))$, and this shows that $g^+ \in R(G)$. Since $g \in R(N,E)$, $g^+ \in R(G,E)$, proving (iii). ■

Notation Before we proceed any further, we introduce the following general notation. Let K be a subgroup of a Lie group G, and let V be a bistable subspace of $R(G)$. We define ${}^K V$ and V^K to be the subspaces

$$\begin{aligned} {}^{K}V &= \{f \in V : f \cdot x = f \text{ for all } x \in K\}, \\ V^{K} &= \{f \in V : x \cdot f = f \text{ for all } x \in K\}. \end{aligned}$$

Note that if K is a normal subgroup of G, then we have ${}^{K}V = V^{K}$.

Theorem 3.3 *Let $G = H \cdot N$ be a semidirect product, where H and N are closed complex analytic subgroups of a complex analytic group G with N normal in G, and let E be a closed normal complex Lie subgroup of G with $E \subset N$. Then*

(i) *the restriction maps*

$$R(G,E)^{N} \to R(H); \ {}^{H}R(G,E) \to R(G,E)_{N}$$

are isomorphisms with inverse maps given by $f \mapsto f^{+}$ (in both cases).

(ii) *$(f,g) \mapsto f^{+}g^{+} : R(H) \times R(G,E)_{N} \to R(G,E)$ induces an isomorphism*

$$R(H) \otimes R(G,E)_{N} \cong R(G,E).$$

In particular, $R(G,E) \cong R(H) \otimes R(N,E)$ canonically, if the restriction morphism $R(G,E) \to R(N,E)$ is surjective.

Proof. (i) Let $\pi : G = H \cdot N \to H$ denote the projection. For $g \in R(H)$, choose an analytic representation ρ of H such that $g \in [\rho]$. Then $\rho \circ \pi$ is a representation of G, which is clearly E-unipotent, and $g^{+} \in [\rho \circ \pi] \subset R(G,E)^{N}$. Now it is easy to check that the restriction map $R(G,E)^{N} \to R(H)$ is an isomorphism with its inverse $g \mapsto g^{+}$. That ${}^{H}R(G,E) \to R(G,E)_{N}$ is an isomorphism follows from Lemma 3.2.

(ii) Let $\mu : R(H) \otimes R(G,E)_{N} \to R(G,E)$ be the morphism of $\mathbb{C}$-algebras, given by $\mu(f \otimes g) = f^{+} \cdot g^{+}$. We first show that μ is surjective. Let γ be the comultiplication on the Hopf algebra $R(G,E)$, and, for $f \in R(G,E)$, write $\gamma(f)$ as

$$\gamma(f) = \sum_{i=1}^{m} f_i \otimes t_i,$$

where $f_i,\ t_i \in R(G,E)$, $1 \le i \le m$. Then for $h \in H$ and $n \in N$, we have

$$f(hn) = \sum_{i=1}^{m} f_i(h)t_i(n). \tag{3.2.3}$$

Let $k_i = (f_i)_H$ and $g_i = (t_i)_N$. Then, from (3.2.3), we get

$$f = \sum_{i=1}^{m} k_i^+ g_i^+ = \mu(\sum_{i=1}^{m} k_i \otimes g_i),$$

proving that μ is surjective.

We next show that μ is an injection. Let $\alpha \in \ker(\mu)$, and write

$$\alpha = \sum_{i=i}^{p} k_i \otimes g_i$$

where each $g_i \in R(G,E)_N$ and $k_i \in R(H)$. Let Y be the $\mathbb{C}$-linear subspace of $R(G,E)_N$ that is spanned by the g_i, and choose a basis $u_1, \cdots, u_q$ of Y and elements $n_1, \cdots, n_q \in N$ so that $u_j(n_i) = \delta_{ij}$ (Lemma 2.7). Then α can be rewritten in the form

$$\alpha = \sum_{i=i}^{q} f_i \otimes u_i,$$

where $f_i \in R(H)$. Then $\sum_{i=1}^{q} f_i^+ u_i^+ = \mu(\alpha) = 0$ implies

$$\sum_{i=1}^{q} f_i(h)u_i(n) = 0$$

for all $h \in H$ and all $n \in N$. With $n = n_j$ in the above equation, we get $f_j(h) = 0$ for all $h \in H$. It follows that all $f_i = 0$, proving $\alpha = 0$. Thus μ is an injection. ∎

3.3 Main Lemma

In this section we prove an important result (Lemma 3.4), which provides a sufficient condition for the extension theorems (Theorem 3.6 and Theorem 3.7) in the next section.

Suppose E is a closed normal complex Lie subgroup of a complex analytic group G, and let $Aut(G,E)$ denote the group of all complex analytic automorphisms of G that leave E stable. Each $\theta \in Aut(G, E)$ induces a unique rational automorphism $\widehat{\theta}$ of the pro-affine algebraic group $A(G, E)$ such that the diagram

$$\begin{CD} G @>{\theta}>> G \\ @V{\tau}VV @VV{\tau}V \\ A(G,E) @>>{\widehat{\theta}}> A(G,E) \end{CD}$$

is commutative (see §2.5).

With this notation in hand, we are now ready to prove our main lemma.

Lemma 3.4 *Let G be a complex analytic group and let E be a closed normal complex Lie subgroup of G with $\dim U(G, E) < \infty$. Let D be a subgroup of $Aut(G, E)$ and assume that D acts as a finite group on G/E. If ρ is an E-unipotent complex analytic representation of G, then the linear span of $[\rho] \circ D$ is finite-dimensional.*

Proof. Let $\tau : G \to A(G, E)$ be the canonical map. Since the representation ρ is E-unipotent, $[\rho] \subset R(G, E)$. We set $R = R(G, E)$, and $U = U(G, E)$. There is a semidirect product decomposition of the pro-affine algebraic group $A(G, E)$

$$A(G, E) = M \cdot U,$$

where M is a maximal reductive subgroup, and this gives a tensor product decomposition of R

$$R = {}^{M}R \cdot R^{U} \cong P(U) \otimes P(M),$$

where $P(U)$ and $P(M)$ are canonically isomorphic with ${}^{M}R$ and R^{U}, respectively (see §B.2). If D_1 denotes the kernel of the action of D on G/E, then D_1 is of finite index in D, and thus our assertion follows as soon as we have shown that the linear span of $[\rho] \circ D_1$ is finite-dimensional. This implies, in particular, that in our proof, we may assume that D itself acts on G/E trivially. For any $\alpha \in A(G, E)$, let $i(\alpha)$ denote the inner automorphism $\beta \mapsto \alpha\beta\alpha^{-1}$ of $A(G, E)$ induced by α. Since $[\rho]$ is bistable under G, it is also bistable under $A(G, E)$ by Lemma 2.11, and thus we get

$$[\rho] \circ i(\alpha) = \alpha^{-1} \cdot [\rho] \cdot \alpha = [\rho]. \tag{3.3.1}$$

Let $\widehat{D} = \{\widehat{\theta} : \theta \in D\}$. For $\theta \in D$, $x \in G$, and $f \in [\rho]$, we have

$$f(\widehat{\theta}(\tau(x))) = \widehat{\theta}(\tau(x))(f)(1) = \tau(\theta(x))(f)(1) = f(\theta(x)).$$

This shows that

$$[\rho] \circ \widehat{D} \circ \tau = [\rho] \circ D$$

and our assertion is therefore equivalent to the statement:

(A) The linear span of $[\rho] \circ \widehat{D}$ is finite-dimensional.

Let P denote the group of all automorphisms of $A(G, E)$ that leave M invariant, and let $\theta \in D$. By the conjugacy of maximal reductive subgroups of the pro-affine algebraic group $A(G, E)$ (Theorem B.8), there is an element $\beta \in U$ such that $i(\beta)(\widehat{\theta}(M)) = M$, and hence $i(\beta)\widehat{\theta} \in P \cap i(U)\widehat{D}$. Thus

$$i(U)\widehat{D} = i(U)D'$$

where $D' = P \cap i(U)\widehat{D}$, and this together with (3.3.1) above yields

$$[\rho] \circ \widehat{D} = [\rho] \circ i(U)\widehat{D} = [\rho] \circ i(U)D' = [\rho] \circ D'.$$

This enables us to replace the statement (A) with the equivalent statement:

(B) The linear span of $[\rho] \circ D'$ is finite-dimensional.

Below, we shall prove (B). For that we first note that the action of $\widehat{D}$ on $A(G, E)/U$ is trivial. In fact, let $\theta \in D$. θ is trivial on G/E by assumption, and clearly $\tau(E) \subset U$. For each $x \in G$, $x^{-1}\theta(x) \in E$, and hence $\tau(x^{-1}\theta(x)) \in U$, i.e., $\tau(\theta(x)) \in \tau(x)U$. Therefore we obtain

$$\widehat{\theta}(\tau(x)U) = \widehat{\theta}(\tau(x))U = \tau(\theta(x))U = \tau(x)U.$$

This shows that each element in $\widehat{D}$ induces the identity map on $\tau(G)U/U$ and hence on its Zariski closure $A(G, E)/U$.

Next we show that the action of D' on M is trivial. Let $\beta \in M$, $\eta \in D' = P \cap i(U)\widehat{D}$, and write $\eta = i(\alpha)\widehat{\theta}$, where $\alpha \in U$ and $\theta \in D$. Since $\widehat{D}$ acts on $A(G, E)/U$ trivially, $\beta^{-1}\widehat{\theta}(\beta) \in U$, and hence

$$\begin{aligned} \beta^{-1}\eta(\beta) &= \beta^{-1}(\alpha\widehat{\theta}(\beta)\alpha^{-1}) \\ &= \beta^{-1}\alpha\beta(\beta^{-1}\widehat{\theta}(\beta))\alpha^{-1} \in U. \end{aligned}$$

On the other hand, $\beta^{-1}\eta(\beta) \in M$, and hence $\beta^{-1}\eta(\beta) \in M \cap U = (1)$, proving that $\eta(\beta) = \beta$. This shows that $D' = 1$ on M.

Since the action of D' on M is trivial, so is the induced action of D' on $R^U \cong P(M)$. We now show that, for any $f \in {}^M R$, $f \circ D'$ has a finite-dimensional linear span in ${}^M R$. We identify ${}^M R$ with $P(U)$, and view U as a closed unipotent complex analytic subgroup of a full complex linear group, $GL(W, \mathbb{C})$ say. The exponential map

$$\exp : \mathcal{L}(U) \to U$$

is a polynomial map and is also a variety isomorphism. Since D' acts on $\mathcal{L}(U)$ linearly via the differential (at 1) of its action on U, $g \circ D'$ spans a finite-dimensional subspace for every polynomial function g on $\mathcal{L}(U)$, and it follows that $f \circ D'$ has a finite-dimensional linear span for any $f \in P(U) = {}^M R$. Since D' fixes the elements of R^U and since $f \circ D'$ has a finite-dimensional linear span for each $f \in {}^M R$, it follows that $[\rho] \circ D'$ has a finite-dimensional linear span. ■

3.4 Extensions of Representations

In this section we state and prove a key result (Theorem 3.7) on extensions of representations.

Lemma 3.5 *Let L be a closed complex Lie subgroup of a complex analytic group G and let E be a closed normal complex Lie subgroup of G such that $E \subset L$. Suppose ρ (resp. σ) is an E-unipotent complex analytic representation of L (resp. G). If σ is an extension of ρ, then $[\rho]$ is contained in the image of $[\sigma]$ under the restriction map $R(G, E) \to R(L, E)$. If every E-unipotent complex analytic representation ρ of L extends to an E-unipotent complex analytic representation σ of G, the restriction map $R(G, E) \to R(L, E)$ is surjective.*

Proof. Let V be the representation space for ρ and let W be the representation space for σ that contains V as an L-stable subspace. Choose a basis $w_1, w_2, \cdots, w_n$ of W such that $w_1, w_2, \cdots, w_m$ $(m \le n)$ is a basis of V, and let $\gamma_1, \gamma_2, \cdots, \gamma_n$ be the basis of the dual space of W which is dual to the basis $w_1, w_2, \cdots, w_n$. Then the restrictions $\gamma_i|_V$ $(1 \le i \le m)$ form a basis of the dual space of V, which is dual to the basis $w_1, w_2, \cdots, w_m$, and the coefficient functions, $g_{i,j} : L \to \mathbb{C}$ of ρ, defined by

$$g_{ij}(y) = \gamma_i(\rho(y)(w_j)),\ 1 \le i \le m,$$

span the space $[\rho]$. Since each g_{ij} is the restriction of the coefficient functions $f_{ij} : G \to \mathbb{C}$ of σ, defined by $f_{ij}(x) = \gamma_i(\sigma(x)(w_j))$, we see that $[\rho]$ is contained in the image of $[\sigma]$ under the restriction map $R(G, E) \to R(L, E)$, proving the first part of the lemma. The second part follows from the first part. ■

Theorem 3.6 *Let G be a complex Lie group, and let H and N be closed complex analytic subgroups of G with N normal in G such that $G = H \cdot N$ is a semidirect product. Let E be a closed normal complex Lie subgroup of G with $E \subset N$. For an E-unipotent complex analytic representation $\rho : N \to GL(W, \mathbb{C})$, the following are equivalent.*

(i) *ρ can be extended to an E-unipotent complex analytic representation of G.*

(ii) *$[\rho] \circ \kappa(H)$ has a finite-dimensional linear span in $R(N, E)$, where $\kappa(h)$, for $h \in H$, denote the automorphism of N given by $n \mapsto hnh^{-1}$.*

Proof. As in §3.2, we use the following notation throughout our proof: Let $R(G, E)_N$ denote the image of $R(G, E)$ in $R(N, E)$ under the restriction map $R(G, E) \to R(N, E)$. Also, for any function $f : N \to \mathbb{C}$ (resp. $f : H \to \mathbb{C}$), define $f^+ : G \to \mathbb{C}$ by $f^+(hn) = f(n)$ (resp. $f^+(hn) = f(h)$) for $n \in N$ and $h \in H$.

(i)⇒ (ii): By Lemma 3.5, we have $[\rho] \subset R(G, E)_N$. Since $[\rho]$ is finite-dimensional, (ii) follows from Lemma 3.2.

(ii)⇒ (i): By Lemma 2.17, the N-module W is embedded as a submodule of the direct sum of finite copies of the N-module $[\rho]$. It is, therefore, enough to show that the N-module $[\rho]$ itself can be embedded into a complex analytic G-module as a sub N-module on which E acts unipotently. Let V denote the linear span of the functions $[\rho] \circ \kappa(H)$. V is then a finite-dimensional and bistable subspace of $R(N, E)$ by Lemma 3.1. Moreover, we have $V^+ \subset R(G, E)$. In fact, $[\rho]^+ \subset R(G, E)$ by Lemma 3.2, and we have

$$(f \circ \kappa(h))^+ = h \cdot f^+ \in R(G, E),$$

for all $h \in H$ and $f \in [\rho]$. Since $([\rho] \circ \kappa(H))^+$ spans V^+, we have $V^+ \subset R(G,E)$. For any $g \in V$, $h \in H$, and $n \in N$, we have

$$n \cdot g^+ = (n \cdot g)^+,$$

and

$$h \cdot g^+ = (g \circ \kappa(h))^+$$

and hence $G \cdot V^+ = V^+$ follows. This shows that V^+ is a G-module, and since $V^+ \subset R(G,E)$, the action of E on V^+ is unipotent. Now the injection

$$f \mapsto f^+ : [\rho] \to V^+$$

is a desired embedding. ∎

Theorem 3.7 *Let G be a complex Lie group, and let H and N be closed complex analytic subgroups of G with N normal in G such that $G = H \cdot N$ is a semidirect product. Let E be a closed normal complex Lie subgroup of G with $E \subset N$. Assume that $\dim U(N,E) < \infty$ and that H induces a finite automorphism group on N/E. Then every E-unipotent complex analytic representation of N can be extended to an E-unipotent complex analytic representation of G. In particular, the restriction map $R(G,E) \to R(N,E)$ is surjective.*

Proof. Let ρ be a E-unipotent complex analytic representation of N. By Lemma 3.4, the linear span of $[\rho] \circ \kappa(H)$ is finite-dimensional, and hence ρ can be extended to a E-unipotent representation of G by Theorem 3.6. The second assertion follows from Lemma 3.5. ∎

Remark 3.8 As for the condition $\dim U(N,E) < \infty$ in Theorem 3.7, we shall first establish that the condition is satisfied if N is simply connected solvable (Proposition 3.15), and this will be generalized to *any* faithfully representable complex analytic group N in Chapter 4 (see Theorem 4.50). ∎

Corollary 3.9 *Under the assumption of Theorem 3.7, let σ denote the projection $G = H \cdot N \to H$. Then the short exact sequence*

$$1 \to N \to G \xrightarrow{\sigma} H \to 1$$

induces a short exact sequence

$$1 \to U(N,E) \to U(G,E) \to U(H) \to 1.$$

Proof. It is enough to show that the morphism $U(N,E) \to U(G,E)$ induced by the inclusion $N \subset G$ is an injection. By Theorem 3.7, the restriction morphism $R(G,E) \to R(N,E)$ is surjective. Since $R(N,E)$ separates the points of $A(N,E)$, we may apply Lemma 2.31 to $S = R(N,E)$ to conclude that $U(N,E) \to U(G,E)$ is an injection. ∎

3.5 Unipotent Analytic Groups

In this section we examine basic properties of unipotent subgroups of a full complex linear group and their unipotent representations. The results from this section are used along with Theorem 3.7 in the proof of Cartan's theorem (Proposition 3.15 of §3.6) on solvable groups.

Let W be a finite-dimensional $\mathbb{C}$-linear space, and let N be a unipotent complex analytic subgroup of $GL(W,\mathbb{C})$. Then N is a nilpotent simply connected subgroup of $GL(W,\mathbb{C})$ (Theorem 2.4). A function on N (resp. on the Lie algebra $\mathcal{L}(N)$) is called a *polynomial function* of degree $\leq k$ if it is the restriction to N (resp. to $\mathcal{L}(N)$) of a polynomial of degree $\leq k$ in complex linear functions on $End_{\mathbb{C}}(W)$.

Lemma 3.10 *Let N be a unipotent complex analytic subgroup of $GL(W,\mathbb{C})$. For any unipotent complex analytic representation of N, $\rho : N \to GL(U,\mathbb{C})$, $[\rho]$ consists of polynomial functions of degree $\leq mn$, where $m = \dim W$, and $n = \dim U$.*

Proof. Let $\exp_W$ and $\exp_U$ denote exponential maps on $GL(W,\mathbb{C})$ and $GL(U,\mathbb{C})$, respectively. Since N is unipotent, $1-x$ is a nilpotent linear transformation on W, and we have, for $x \in N$,

$$\log_W(x) = -\sum_{i=1}^{m} \frac{1}{i}(1-x)^i. \tag{3.5.1}$$

Also, for $x \in N$, we have

$$\begin{aligned} \rho(x) &= \rho(\exp_W \circ \log_W(x)) \\ &= \exp_U \circ d\rho(\log_W(x)). \end{aligned} \tag{3.5.2}$$

If $\mathfrak{n} = \mathcal{L}(N)$, every $z \in d\rho(\mathfrak{n})$ is a nilpotent linear transformation of U, and we have

$$\exp_U(z) = \sum_{i=0}^{n} \frac{1}{i!} z^i. \tag{3.5.3}$$

Let $f \in [\rho]$, and we write it in the form $f = \lambda \circ \rho$, where λ is a linear function on $End_{\mathbb{C}}(U)$. Then it is now clear from the formulas (3.5.1), (3.5.2), and (3.5.3) above that f is a polynomial function of degree $\leq mn$. ■

Corollary 3.11 *If N is a unipotent complex analytic subgroup of a general linear group $GL(W, \mathbb{C})$, then every automorphism η of N is a polynomial map of degree $\leq m^2$, where $m = \dim W$.*

Proof. The assertion follows from Lemma 3.10 by letting $U = W$ and regarding the automorphism η of N as a representation of N on W. ■

Proposition 3.12 *Let W be a finite-dimensional $\mathbb{C}$-linear space. Any unipotent complex analytic subgroup of $GL(W, \mathbb{C})$ is an algebraic subgroup of $GL(W, \mathbb{C})$.*

Proof. Let N^* denote the Zariski closure of N in $GL(W, \mathbb{C})$, and let $y \in N^*$. Put $\mathcal{A} = P(GL(W, \mathbb{C}))$, the polynomial algebra of $GL(W, \mathbb{C})$. If $f \in \mathcal{A}$ annihilates the elements of N, then $f(y) = 0$. Define $y' : \mathcal{A}_N \to \mathbb{C}$ by $y'(f_N) = f(y)$ for all $f \in \mathcal{A}$. Since y is in the Zariski closure of N, y' is well defined, and is a $\mathbb{C}$-algebra homomorphism. Let $\mathfrak{n} = \mathcal{L}(N)$, and let $\mathfrak{n}^\circ = Hom_{\mathbb{C}}(\mathfrak{n}, \mathbb{C})$. For any $\lambda \in \mathfrak{n}^\circ$, the composite $\lambda \circ \log : N \to \mathbb{C}$ is a polynomial function on N, i.e., $\lambda \circ \log \in \mathcal{A}_N$, by what we have seen in the proof of Lemma 3.10 (see the formula (3.5.1)). Now, the map

$$\lambda \mapsto y'(\lambda \circ \log) : \mathfrak{n}^\circ \to \mathbb{C}$$

is $\mathbb{C}$-linear, and hence there exists $z \in \mathfrak{n}$ such that $y'(\lambda \circ \log) = \lambda(z)$ for all $\lambda \in \mathfrak{n}^\circ$. If $x = \exp(z)$, then we have

$$y'(\lambda \circ \log) = (\lambda \circ \log)(x) \tag{3.5.4}$$

for all $\lambda \in \mathfrak{n}^\circ$.

Let $g \in \mathcal{A}_N$. Using the same argument as in the proof of Lemma 3.10 (see (3.5.3)), we see that $g \circ \exp$ is a polynomial function on $\mathfrak{n}$, and therefore $g \circ \exp$ is a polynomial in linear functionals $\lambda \in \mathfrak{n}^\circ$, i.e., g is a polynomial in the elements $\lambda \circ \log$. Since y' is an algebra homomorphism, the identity (3.5.4) yields $y'(g) = g(x)$. This shows that $f(y) = y'(f) = f(x)$ for all $f \in \mathcal{A}$. Since the functions in $\mathcal{A}$ separate the points of $GL(W, \mathbb{C})$, $y = x \in N$, and this proves $N^* = N$. ■

Corollary 3.13 *A unipotent complex analytic subgroup of* $GL(n, \mathbb{C})$ *is closed and simply connected.*

Proof. As an algebraic subgroup of $GL(n, \mathbb{C})$, a unipotent complex analytic subgroup G of $GL(n, \mathbb{C})$ is closed. Also, by Theorem 2.4, G is conjugate to a subgroup of $U(n, \mathbb{C})$. Since $U(n, \mathbb{C})$ is simply connected, so are its closed analytic subgroups. Thus G is simply connected. ■

In light of Proposition 3.12, we next determine the polynomial algebra of the algebraic group N.

Theorem 3.14 *Let* N *be a unipotent complex analytic subgroup of a full linear group* $GL(W, \mathbb{C})$*. Then* $P(N) = R(N, N)$*.*

Proof. Clearly $P(N) \subset R(N, N)$. To show $P(N) \supset R(N, N)$, let $f \in R(N, N)$, and express $f = \lambda \circ \rho$, where $\rho : N \to GL(U, \mathbb{C})$ is a unipotent complex analytic representation, and where λ is a $\mathbb{C}$-linear functional on $End(U, \mathbb{C})$. By Lemma 3.10, f is a polynomial function, and $f \in P(N)$ follows. ■

3.6 Application to Solvable Groups

As an application of the extension theorem (Theorem 3.7), we prove a result of Cartan on the existence of faithful representations of simply connected solvable groups. We also examine the dimension of the unipotent hull of solvable groups, which will be later generalized to faithfully representable analytic groups (see Theorem 4.50).

Proposition 3.15 *Let* G *be a simply connected, solvable complex analytic group, and let* M *be a closed, nilpotent, normal complex analytic subgroup of* G *such that* G/M *is abelian. Then there is a faithful complex analytic representation of* G *which is unipotent on* M*. In this case, we have*

$$\dim U(G, M) = \dim G.$$

Proof. Let $\mathfrak{m} = \mathcal{L}(M)$, and let τ be a faithful complex analytic unipotent representation of the simply connected analytic group M. To see that such a representation exists, we first choose, by Ado's theorem ([6], Th. 5, p. 153), a finite-dimensional faithful nilpotent representation of $\mathfrak{m}$:

$$\mathfrak{m} \to \mathfrak{gl}(W, \mathbb{C}),$$

and note that this representation is the differential of a unipotent complex analytic representation, say

$$\tau : M \to GL(W, \mathbb{C}).$$

The connected unipotent group $\tau(M)$ is simply connected, and $\ker(\tau)$ is therefore connected. On the other hand, $\ker(\tau)$ is discrete because the differential of τ is faithful, and hence $\ker(\tau) = (1)$, proving that τ is faithful. By Proposition 3.12, $\tau(M)$ is a unipotent algebraic subgroup of $GL(W, \mathbb{C})$.

The complex vector group G/M contains complex 1-parameter subgroups $P_1, \cdots, P_m$ so that G is obtained as successive semidirect products

$$G = P_m \cdots P_1 \cdot M.$$

We let $G_0 = M$ and $\sigma_0 = \tau$, and for $1 \leq i \leq m$ define

$$G_i = P_i \cdots P_1 \cdot M.$$

The complex analytic isomorphism $M \cong \tau(M)$ yields

$$R(M, M) \cong R(\tau(M), \tau(M)).$$

On the other hand,

$$P(\tau(M)) = R(\tau(M), \tau(M))$$

by Theorem 3.14, and hence the canonical map

$$\tau(M) \to A(\tau(M), \tau(M))$$

is an isomorphism. Consequently, $A(\tau(M), \tau(M))$ is unipotent, and we have an isomorphism

$$\tau(M) \cong A(\tau(M), \tau(M)) = U(\tau(M), \tau(M)) \cong U(M, M).$$

This, in particular, shows that

$$\dim U(M, M) = \dim \tau(M) = \dim M < \infty. \tag{3.6.1}$$

We now prove, by induction on i, that the representation σ_0 extends to an M-unipotent complex analytic representation σ_i of G_i and that

$$\dim U(G_i, M) = \dim M + i$$

for $0 \leq i \leq m$. The case for $i = 0$ is established in (3.6.1). Assume that σ_0 has been already extended to an M-unipotent complex analytic representation σ_i of the group G_i, and that

$$\dim U(G_i, M) = \dim M + i < \infty,$$

where $0 \leq i \leq m - 1$, and consider the exact sequence of solvable groups

$$1 \to G_i \to G_{i+1} \to P_{i+1} \to 1. \qquad (3.6.2)$$

G is a semidirect product $G_{i+1} = P_{i+1} \cdot G_i$, and since G/M is abelian, the action of P_{i+1} on G_i/M is trivial. Hence by Theorem 3.7 we obtain an M-unipotent complex analytic representation σ_{i+1} of the group G_{i+1} which extends σ_i, and the restriction map

$$R(G_{i+1}, M) \to R(G_i, M)$$

is surjective. The exact sequence (3.6.2) induces the exact sequence

$$1 \to U(G_i, M) \to U(G_{i+1}, M) \to U(P_{i+1}) \to 1 \qquad (3.6.3)$$

by Corollary 3.9. Since $P_{i+1} \cong \mathbb{C}$, we have $\dim U(P_{i+1}) = 1$ (see Example 2.28), and it follows from the exact sequence (3.6.3) and from the induction hypothesis that

$$\dim U(G_{i+1}, M) = \dim U(G_i, M) + 1 = \dim M + (i + 1) < \infty.$$

This completes the proof of the assertion at the $(i+1)$-th step. If we let $\sigma = \sigma_m$, σ is an M-unipotent complex analytic representation of G, which extends τ, and

$$\dim U(G, M) = \dim U(G_m, M) = \dim M + m = \dim G.$$

To complete our proof of the proposition, choose any faithful complex analytic representation of the vector group G/M, and compose it with the natural morphism $G \to G/M$ to obtain a complex analytic representation ρ of G with kernel M. The direct sum of σ and ρ is a desired faithful M-unipotent complex analytic representation. ■

Chapter 4

The Structure of Complex Lie Groups

This chapter deals with the general structure theory of complex Lie groups. Sections 4.2 and 4.3 are devoted to semisimple complex groups, and the decomposition theorem of complex groups in §4.7 follows the discussions on reductive groups ([2], [9], [10], [13]).

If a complex (resp. real) analytic group G admits a faithful finite-dimensional complex (resp. real) analytic representation, then we shall simply say that G is *faithfully representable.*

4.1 Abelian Complex Analytic Groups

Structure of Abelian Complex Analytic Groups

Proposition 4.1 *Let G be an abelian complex analytic group, and let T be the maximum compact subgroup of G. Then $G = T^* \times U$, where T^* denotes the smallest complex analytic subgroup of G that contains T, and U is a complex vector subgroup. If* every *complex analytic representation of G is semisimple, then $G = T^*$.*

Proof. If we view G as an abelian real analytic group, G/T is a real vector group, and hence the analytic subgroup T^*/T is closed in G/T. This shows that T^* is also closed in G, and if we choose a complex analytic subgroup U of G such that $\mathcal{L}(G) = \mathcal{L}(T^*) \oplus \mathcal{L}(U)$, then $G = T^*U$. Consider the canonical homomorphism $\pi : U \to G/T^*$. It is a covering morphism of the analytic group G/T^*. Since G/T^* is a quotient group of the real vector group G/T by the vector subgroup T^*/T, G/T^* is also a real vector group and hence is simply connected. Thus π is an isomorphism, and $T^* \cap U = (1)$ follows. This shows that $G = T^* \cdot U$ is a direct product, and U is a complex vector group.

For the second assertion, suppose U is *not* trivial. Then the vector group U has a faithful complex analytic unipotent representation (see Example 2.6, (ii)), and the composite of this with the projection $G \to U$ yields a nontrivial unipotent complex analytic representation of G. ∎

A closed subgroup T of a complex Lie group G is called *full* in G if $\mathcal{L}(T)$ spans $\mathcal{L}(G)$ over $\mathbb{C}$ and $G = TG_0$, where G_0 is the identity component of G. (We note that T is a real Lie subgroup as a closed subgroup of a Lie group.) Proposition 4.1 thus states that if every complex analytic representation of an abelian complex analytic group G is semisimple, then the maximal compact subgroup T of G is full in G. Later we shall see that every complex analytic representation of a faithfully representable complex analytic group G is semisimple if and only if G has a full compact subgroup (see Proposition 4.22).

Theorem 4.2 *Let G be an abelian complex analytic group, and let T be the maximum compact subgroup of G. Assume that T is full in G and that $\dim_{\mathbb{R}} T = \dim_{\mathbb{C}} G$. Then every (real) analytic representation*

$$\rho : T \to GL(V, \mathbb{C})$$

extends uniquely to a complex analytic representation

$$\widetilde{\rho} : G \to GL(V, \mathbb{C}).$$

Moreover, if ρ is faithful, then so is $\widetilde{\rho}$.

Proof. The uniqueness of $\widetilde{\rho}$ follows from the fullness of T in G. We now prove the existence of an extension $\widetilde{\rho}$. Let $\mathfrak{g} = \mathcal{L}(G)$. Since T is full in G, the canonical morphism of Lie algebras

$$\mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}(T) \to \mathfrak{g}$$

is a surjection, and hence an isomorphism because of the condition $\dim_{\mathbb{R}} T = \dim_{\mathbb{C}} G$. The abelian real analytic group G can be written as a direct product $G = T \cdot U$, where U is a real vector subgroup of G, and hence $\mathfrak{g} = \mathcal{L}(T) \oplus \mathcal{L}(U)$. If we view the exponential map

$$\exp_G : \mathfrak{g} \to G$$

as a surjective morphism of the complex vector group $\mathfrak{g}$ onto G, and its restriction to $\mathcal{L}(T)$

$$\exp_T : \mathcal{L}(T) \to T$$

as a morphism of real analytic groups, then $\exp_G$ is an injection, when restricted to $\mathcal{L}(U)$, and hence we have

$$\ker(\exp_G) = \ker(\exp_T). \tag{4.1.1}$$

Since T is a compact group, the representation ρ is semisimple, and each irreducible representation of T is 1-dimensional. Therefore we may decompose the representation space V into 1-dimensional T-stable linear subspaces $V = V_1 \oplus \cdots \oplus V_n$, and the differential $d\rho$ of ρ determines $\mathbb{R}$-linear functions

$$f_k : \mathcal{L}(T) \to \mathbb{R}$$

such that, for each $X \in \mathcal{L}(T)$, $d\rho(X)$ acts on V_k as the multiplication by $\sqrt{-1} f_k(X)$. Let g_k denote the $\mathbb{C}$-linear function

$$1_{\mathbb{C}} \otimes f_k : \mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}(T) \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R} = \mathbb{C}.$$

Fixing a suitable basis for V provided by the above decomposition of V, we may identify $GL(V, \mathbb{C})$ with $GL(n, \mathbb{C})$, and $\mathfrak{gl}(V, \mathbb{C})$ with $\mathfrak{gl}(n, \mathbb{C})$ so that, for each $X \in \mathcal{L}(T)$, we have

$$d\rho(X) = diag(\sqrt{-1} f_1(X), \cdots, \sqrt{-1} f_n(X)).$$

The complex representation $\sigma : \mathfrak{g} \to \mathfrak{gl}(n, \mathbb{C})$, defined by

$$\sigma(X) = diag(\sqrt{-1} g_1(X), \cdots, \sqrt{-1} g_n(X)),$$

is clearly an extension of the representation $d\rho$. If exp denotes the exponential map $\exp : \mathfrak{gl}(n, \mathbb{C}) \to GL(n, \mathbb{C})$, the composite map

$$\exp \circ \sigma : \mathfrak{g} \to GL(n, \mathbb{C})$$

is then a complex analytic representation of the complex vector group $\mathfrak{g}$, and its restriction to $\mathcal{L}(T)$ coincides with $\exp \circ d\rho = \rho \circ \exp_T$. Hence $\exp \circ \rho$ maps $\ker(\exp_T)$ to 1. Since $d\rho = \sigma$ on $\mathcal{L}(T)$, (4.1.1) implies that $\exp \circ \sigma$ maps $\ker(\exp_G)$ to 1, and hence it induces a unique complex analytic representation $\widetilde{\rho} : G \to GL(n, \mathbb{C})$ so that $\widetilde{\rho} \circ \exp_G = \exp \circ \sigma$. Then $\widetilde{\rho} \circ \exp_G = \exp \circ d\rho = \rho \circ \exp_G$ on $\mathcal{L}(T)$, from which we get $\widetilde{\rho} = \rho$ on T.

Now we show that if ρ is faithful, then so is $\widetilde{\rho}$. Let $z \in \ker(\widetilde{\rho})$, and let $Z \in \mathfrak{g}$ such that $z = \exp_G(Z)$. Writing $Z = Z_1 + \sqrt{-1}Z_2$ with $Z_1, Z_2 \in \mathcal{L}(T)$, we have

$$\begin{aligned} I_n &= \widetilde{\rho}(z) = (\exp \circ \sigma)(Z) \\ &= \exp(\sigma(Z_1) + \sqrt{-1}\sigma(Z_2)) \\ &= diag(e^{\sqrt{-1}f_1(Z_1) - f_1(Z_2)}, \cdots, e^{\sqrt{-1}f_n(Z_1) - f_n(Z_2)}). \end{aligned}$$

Hence $e^{\sqrt{-1}f_k(Z_1) - f_k(Z_2)} = 1$, and this implies $f_k(Z_2) = 0$ for all k, obtaining $d\rho(Z_2) = 0$. But $d\rho$ is faithful on $\mathcal{L}(T)$, and hence $Z_2 = 0$, and we have $Z \in \mathcal{L}(T)$ and $z \in T$. Since $\widetilde{\rho} = \rho$ on T, and since ρ is trivial on T, we have $z = 1$, proving that $\widetilde{\rho}$ is faithful. ■

Complex Tori Recall (from §1.3) that any complex analytic group isomorphic with $(\mathbb{C}^*)^n$ is called a *complex torus*. The maximum compact subgroup of an n-dimensional complex torus has the real dimension n, and we shall show below that this property actually characterizes complex tori (Corollary 4.4). We need the following result which we shall generalize to all complex analytic groups. (See Theorem 4.30.)

Proposition 4.3 *Let G be an abelian complex analytic group, and let T be a compact full (real) analytic subgroup of G. If a complex analytic representation $\rho : G \to GL(V, \mathbb{C})$ is faithful on T, then ρ is faithful, and G is an n-dimensional complex torus, where $n = dim_{\mathbb{R}} T$.*

Proof. We first identify the (real) torus T with the subgroup of the complex analytic group $D = (\mathbb{C}^*)^n$. Then T is the maximal compact subgroup of D, and T is full in D. Hence, by Theorem 4.2, there is a faithful complex analytic representation $\tau : D \to GL(V, \mathbb{C})$ such that $\tau = \rho$ on T.

Since the complex analytic representations τ and ρ agree on T, and since T is full in both D and G, we have $\tau(D) = \rho(G)$. We consider the map $\tau^{-1} \circ \rho : G \to D$. It is a surjective morphism, and $\tau^{-1} \circ \rho = 1$ on T. Hence $\tau^{-1} \circ \rho$ induces a surjective morphism $G/T \to D/T$, which implies

$$\dim_{\mathbb{R}}(D/T) \leq \dim_{\mathbb{R}}(G/T).$$

On the other hand, the fullness of T in G implies that the canonical map

$$\mathcal{L}(D) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}(T) \to \mathfrak{g}$$

is surjective, and hence $\dim_{\mathbb{R}} D \geq \dim_{\mathbb{R}} G$. This implies

$$\dim_{\mathbb{R}}(D/T) \geq \dim_{\mathbb{R}}(G/T).$$

We now have $\dim_{\mathbb{R}}(D/T) = \dim_{R}(G/T)$, and the map $G/T \to D/T$ is a covering morphism, which is an isomorphism, because D/T is simply connected. This readily implies that $\tau^{-1} \circ \rho : G \to D$ is an isomorphism. This shows that ρ is faithful, and $G \cong D = (\mathbb{C}^*)^n$. ■

Corollary 4.4 *Let T be the maximum compact subgroup of an abelian complex analytic group G, and suppose that T is full in G. Then these are equivalent.*

(i) $\dim_{\mathbb{R}} T = \dim_{\mathbb{C}} G$*;*

(ii) *G is faithfully representable;*

(iii) *G is a complex torus.*

Proof. (ii)⇒(iii) is a special case of Proposition 4.3, and (iii)⇒(i) is trivial. Since the compact group T has a faithful real analytic representation, (i)⇒(ii) follows from Theorem 4.2. ■

As an immediate consequence of Corollary 4.4, we have

Corollary 4.5 *A nontrivial compact complex analytic group G never admits a faithful analytic representation.* ■

4.2 Decomposition of the Adjoint Group

The aim of this section is to study the decomposition of the adjoint group of a semisimple Lie algebra (Theorem 4.15), which will be used in §4.3 for the global decomposition of semisimple Lie groups. All Lie algebras are assumed to be finite-dimensional, unless stated otherwise.

The Adjoint Group Let $\mathfrak{g}$ be a real Lie algebra. The adjoint representation ad of $\mathfrak{g}$ maps each $X \in \mathfrak{g}$ to a derivation of $\mathfrak{g}$, and hence $ad(\mathfrak{g})$ is a Lie subalgebra of $Der(\mathfrak{g})$. $Der(\mathfrak{g})$ is the Lie algebra of the closed real Lie subgroup $Aut(\mathfrak{g})$ of $GL(\mathfrak{g}, \mathbb{R})$, and the analytic subgroup of $Aut(\mathfrak{g})$ that corresponds to the subalgebra $ad(\mathfrak{g})$ is called the *adjoint group* of $\mathfrak{g}$, and is denoted by $Int(\mathfrak{g})$. $Int(\mathfrak{g})$ is generated by the one-parameter subgroups $\exp t\ ad(X)$ in $Aut(\mathfrak{g})$, where $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Note that if G is a real analytic group with Lie algebra $\mathfrak{g}$, then $Int(\mathfrak{g}) = Ad(G)$, where $Ad : G \to GL(\mathfrak{g}, \mathbb{R})$ denotes the adjoint representation of G. Also note that if $\mathfrak{g}$ is semisimple, then $Der(\mathfrak{g}) = ad(\mathfrak{g})$ (Theorem A.13), and hence $Int(\mathfrak{g}) = Aut_0(\mathfrak{g})$.

A real Lie algebra $\mathfrak{g}$ is said to be *compact* if the group $Int(\mathfrak{g})$ is compact. The Lie algebra of a compact analytic group G is compact. On the other hand, every compact real Lie algebra is isomorphic with the Lie algebra of a compact analytic group.

We have the following general result on compact groups.

Lemma 4.6 *Let G be a compact group, and let V be a continuous G-module over $\mathbb{R}$ (resp. $\mathbb{C}$). V admits a G-invariant inner product (resp. Hermitian inner product) $< , >$, i.e.,*

$$< x \cdot u, x \cdot v > = < u, v >$$

for all $u, v \in V$ and $x \in G$.

Proof. We prove the lemma for the case in which V is a complex G-module. Choose any Hermitian inner product $(,)$ on V. Given $u, v \in V$, the function $f_{u,v} : G \to \mathbb{C}$ defined by $f_{u,v}(x) = (x \cdot u, x \cdot v)$ is continuous. Define

$$< u, v > = \int_G f_{u,v}(x) dx,$$

where dx denotes a Haar measure on the compact group G normalized so that $\int_G dx = 1$. Then $< \, , \, >$ is easily seen to be a Hermitian inner product. We now show that $< \, , \, >$ is G-invariant. For $u, v \in V$ and $y \in G$, we have

$$f_{y \cdot u, y \cdot v} = y \cdot f_{u,v},$$

and hence

$$\begin{aligned} < y \cdot u, y \cdot v > &= \int_G f_{y \cdot u, y \cdot v}(x) dx \\ &= \int_G y \cdot f_{u,v}(x) dx \\ &= \int_G f_{u,v}(x) dx \\ &= < u, v > \end{aligned}$$

by the invariance of the Haar measure. ■

Proposition 4.7 *Let G be a real analytic group, and view the Lie algebra $\mathfrak{g}$ of G as a G-module under the adjoint action*

$$(x, X) \mapsto x \cdot X = Ad(x)(X) : G \times \mathfrak{g} \to \mathfrak{g}.$$

An inner product $< \, , \, >$ on the G-module $\mathfrak{g}$ is G-invariant if and only if it is an invariant bilinear form on $\mathfrak{g}$ (see §A.1)*, i.e.,*

$$< [Z, X], Y > + < X, [Z, Y] > = 0 \quad \forall X, Y, Z \in \mathfrak{g}. \tag{4.2.1}$$

Proof. For $A \in \mathfrak{gl}(\mathfrak{g}, \mathbb{R})$, let e^A denote the usual exponential of A. For $X, Z \in \mathfrak{g}$ and $t \in \mathbb{R}$, we have

$$(\exp tZ) \cdot X = e^{t\, ad(Z)}(X),$$

and hence

$$\frac{d}{dt}((\exp tZ) \cdot X) = [Z, (\exp tZ) \cdot X] \tag{4.2.2}$$

Assume that $< \, , \, >$ is G-invariant. Then for $X, Y, Z \in \mathfrak{g}$,

$$< (\exp tZ) \cdot X, (\exp tZ) \cdot Y > = < X, Y >, \; \forall t \in \mathbb{R}.$$

Differentiating this with respect to t at $t = 0$, we obtain (4.2.1).

Suppose now that the inner product $<\ ,\ >$ satisfies (4.2.1). For $X, Y, Z \in \mathfrak{g}$ and $t \in \mathbb{R}$, we have

$$\begin{aligned}&\frac{d}{dt} < (\exp tZ) \cdot X, (\exp tZ) \cdot Y > \\ &= < [Z, (\exp tZ) \cdot X], (\exp tZ) \cdot Y > + < (\exp tZ) \cdot X, [Z, (\exp tZ) \cdot Y] > \\ &= 0\end{aligned}$$

by (4.2.1) and (4.2.2). Consequently, the map

$$t \mapsto < (\exp tZ) \cdot X, (\exp tZ) \cdot Y >: \mathbb{R} \to \mathbb{R}$$

is a constant function with the value $< X, Y >$, and since the one-parameter subgroups of G generate the entire group G, this readily implies that $<\ ,\ >$ is G-invariant. ■

Now we study compact Lie algebras. For that, first recall the following theorem. (See, e.g., Corollary 1, [26], Exposé 22°.)

Theorem 4.8 (Weyl) *The universal covering group of a compact semisimple analytic group is compact.* ■

Theorem 4.9 *Let $\mathfrak{g}$ be a real Lie algebra. Then the following are equivalent.*

(i) $\mathfrak{g}$ *is compact.*

(ii) $[\mathfrak{g}, \mathfrak{g}]$ *is compact semisimple, and* $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$, *where* $\mathfrak{z}$ *is the center of* $\mathfrak{g}$.

Moreover, if $\mathfrak{g}$ *is semisimple, then the conditions above are equivalent to*

(iii) *The Killing form on* $\mathfrak{g}$ *is negative definite.*

Proof. Let κ denote the Killing form on $\mathfrak{g}$.

(i)⇒ (ii): Assume $\mathfrak{g}$ is compact. Thus $Int(\mathfrak{g})$ is compact, and by Lemma 4.6, the $\mathbb{R}$-linear space $\mathfrak{g}$ admits an $Int(\mathfrak{g})$-invariant inner product $<\ ,\ >$, which is $ad(\mathfrak{g})$-invariant by Proposition 4.7. For $Y \in \mathfrak{g}$, let $(a_{i,j}(Y))$ denote the matrix of $ad(Y)$ with respect to a fixed orthonormal basis $X_1, \cdots, X_n$ of $\mathfrak{g}$. From

$$< ad(Y)(X_i), X_j > + < X_i, ad(Y)(X_j) >= 0$$

for all i, j, we see that the matrix of $ad(Y)$ is skew symmetric, and hence

$$\kappa(Y,Y) = Tr(ad(Y)^2) = -\sum_{i,j} a_{i,j}(Y)^2 \leq 0,$$

where the equality holds if and only if $ad(Y) = 0$, i.e., $Z \in \mathfrak{z}$. Let $\mathfrak{a}$ denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{g}$. Then $\mathfrak{a}$ is an ideal, and $\kappa(Y,Y) < 0$ for all nonzero $Y \in \mathfrak{a}$. Since $\kappa_{\mathfrak{a}} = \kappa|_{\mathfrak{a}\times\mathfrak{a}}$ by Lemma A.1, we see that $\kappa_{\mathfrak{a}}$ is negative definite. It follows that $\mathfrak{a}$ is a compact semisimple Lie algebra, and $\mathfrak{a} = [\mathfrak{g}, \mathfrak{g}]$.

(ii)$\Rightarrow$ (i): Let $\mathfrak{g}'$ denote $[\mathfrak{g}, \mathfrak{g}]$, and assume $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ (direct product), where $\mathfrak{g}'$ is compact semisimple. Then $ad_{\mathfrak{g}}(\mathfrak{g}) = ad_{\mathfrak{g}}(\mathfrak{g}')$, and the restriction map

$$ad_{\mathfrak{g}}(Y) \mapsto ad_{\mathfrak{g}'}(Y) : ad_{\mathfrak{g}}(\mathfrak{g}') \to ad_{\mathfrak{g}'}(\mathfrak{g}')$$

is an isomorphism of Lie algebras. Hence the restriction map

$$Int(\mathfrak{g}) \to Int(\mathfrak{g}')$$

is a covering morphism of the $Int(\mathfrak{g}')$, which is compact and semisimple by the assumption. $Int(\mathfrak{g})$ is therefore compact by Theorem 4.8, i.e., $\mathfrak{g}$ is compact, and (i) is proved.

(iii)$\Rightarrow$ (i): Suppose $\mathfrak{g}$ is semisimple and its Killing form κ is negative definite. Since κ is invariant on $\mathfrak{g}$ (§A.1), it is $Int(\mathfrak{g})$-invariant by Proposition 4.7. Thus $Int(\mathfrak{g})$ is a closed subgroup of the compact group $O(\kappa)$, the subgroup of $GL(\mathfrak{g}, \mathbb{R})$ consisting of all transformations which leave κ invariant, and hence $Int(\mathfrak{g})$ is compact, proving (iii)$\Rightarrow$ (i).

The proof for (i)$\Rightarrow$ (iii) is contained in the proof of (i)$\Rightarrow$ (ii). ∎

Theorem 4.8 also provides the following consequence of Theorem 4.9.

Theorem 4.10 *Let G be a semisimple real analytic group with Lie algebra $\mathfrak{g}$. Then G is compact if and only if $\mathfrak{g}$ is compact.*

Proof. If G is compact, then so is $Int(\mathfrak{g}) = Ad(G)$, and hence $\mathfrak{g}$ is compact. Conversely, suppose $\mathfrak{g}$ is compact. The adjoint map $Ad : G \to Int(\mathfrak{g}) = Ad(G)$ is then a covering morphism of the compact group $Int(\mathfrak{g})$, and hence G is compact by Theorem 4.8. ∎

Real Forms of a Complex Lie Algebra For a complex Lie algebra $\mathfrak{g}$, let $\mathfrak{g}_{\mathbb{R}}$ denote the real Lie algebra obtained from $\mathfrak{g}$ by restricting the action of $\mathbb{C}$ to $\mathbb{R}$. Recall (§1.5) that a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}_{\mathbb{R}}$ is called a *real form* of $\mathfrak{g}$ if the canonical $\mathbb{R}$-bilinear map $(c, X) \mapsto cX : \mathbb{C} \times \mathfrak{g}_0 \to \mathfrak{g}$ induces an isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_0 \cong \mathfrak{g}$ of complex Lie algebras, or equivalently, if $\mathfrak{g} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$. In this case, the conjugation

$$X + \sqrt{-1}Y \mapsto X - \sqrt{-1}Y$$

for $X, Y \in \mathfrak{g}_0$ is an isomorphism of the real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ with itself. A compact Lie algebra which is also a real form of a complex Lie algebra is called a *compact real form* of $\mathfrak{g}$.

For the proof of the following on the existence of a compact real form of a complex Lie algebra, see, e.g., [9], Chapter III.

Theorem 4.11 *Every semisimple complex Lie algebra $\mathfrak{g}$ admits a compact real form. Any two compact real forms of $\mathfrak{g}$ are conjugate via an element of the adjoint group of $\mathfrak{g}$.* ■

Let $\mathfrak{g}_0$ be a real form of a complex Lie algebra $\mathfrak{g}$, and let κ_0, κ, and $\kappa_{\mathbb{R}}$ denote the Killing forms of $\mathfrak{g}_0$, $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{R}}$, respectively. Then we have

$$\kappa_0(X, Y) = \kappa(X, Y), \ X, Y \in \mathfrak{g}_0; \tag{4.2.3}$$

$$\kappa_{\mathbb{R}}(X, Y) = 2Re(\kappa(X, Y)), \ X, Y \in \mathfrak{g}_{\mathbb{R}}. \tag{4.2.4}$$

In light of the semisimplicity criterion (Theorem A.12), we deduce the following from (4.2.3) and (4.2.4).

Proposition 4.12 *Lie algebras $\mathfrak{g}_0$, $\mathfrak{g}$, and $\mathfrak{g}_{\mathbb{R}}$ are all semisimple if and only if one of them is so.* ■

The negative definiteness of the killing form on a semisimple Lie algebra (Theorem 4.9) yields

Proposition 4.13 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and let $\mathfrak{g}_0$ be a compact real form of $\mathfrak{g}$. If θ denotes the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$, define*

$$B_\theta(X, Y) = -\kappa_{\mathfrak{g}}(X, \theta(Y))$$

for $X, Y \in \mathfrak{g}$. Then B_θ is a positive definite Hermitian form on $\mathfrak{g}$. ■

Given a complex semisimple Lie algebra $\mathfrak{g}$, B_θ will denote the positive definite Hermitian form on $\mathfrak{g}$, which is associated with a (fixed) compact real form $\mathfrak{g}_0$ as in Proposition 4.13.

Theorem 4.14 *For a complex Lie algebra $\mathfrak{g}$, $Aut(\mathfrak{g})$ is an algebraic subgroup of $GL(\mathfrak{g}, \mathbb{C})$, which is self-adjoint with respect to B_θ.*

Proof. We put $B = B_\theta$. Let $X_1, \cdots, X_n$ be a basis of $\mathfrak{g}$ over $\mathbb{C}$, and let $c_{i,j}^k \in \mathbb{C}$ be the structural constants of $\mathfrak{g}$, which are defined by

$$[X_i, X_j] = \sum_{k=1}^{n} c_{i,j}^k X_k, \ (1 \leq i, j \leq n).$$

Then $\alpha \in GL(\mathfrak{g}, \mathbb{C})$ is in $Aut(\mathfrak{g})$ if and only if α satisfies

$$\alpha([X_i, X_j]) - [\alpha(X_i), \alpha(X_j)] = 0, 1 \leq i, j \leq n. \tag{4.2.5}$$

Let

$$\alpha(X_j) = \sum_k a_{k,j}(\alpha) X_k.$$

Then the functions $\alpha \mapsto a_{k,j}(\alpha)$ are coefficient functions, and the identity (4.2.5) holds if and only if the polynomials

$$\sum_{k,m} c_{k,m}^p a_{k,i} a_{m,j} - \sum_{i,j} c_{i,j}^k a_{p,k}, \ 1 \leq i, j, p \leq n$$

vanish on $Aut(\mathfrak{g})$. This shows that $Aut(\mathfrak{g})$ is an algebraic subgroup.

Next we show that if $\alpha \in Aut(\mathfrak{g})$, then $\alpha^* \in Aut(\mathfrak{g})$. We first establish the following:

$$B([X, Y], Z) = -B(Y, [\theta(X), Z]), \ X, Y, Z \in \mathfrak{g}; \tag{4.2.6}$$
$$\theta \circ \beta \circ \theta \circ \beta^* = 1, \ \beta \in Aut(\mathfrak{g}). \tag{4.2.7}$$

In fact,

$$\begin{aligned} B([X, Y], Z) &= -\kappa_{\mathfrak{g}}([X, Y], \theta(Z)) \\ &= \kappa_{\mathfrak{g}}(Y, [X, \theta(Z)]) \\ &= \kappa_{\mathfrak{g}}(Y, \theta[\theta(X), Z]) \\ &= B(Y, [\theta(X), Z]), \end{aligned}$$

proving (4.2.6).

For (4.2.7), use $\beta \circ ad(X) = ad(\beta(X)) \circ \beta$, $X \in \mathfrak{g}$, to get

$$ad(\beta(X)) \circ ad(Y) = \beta \circ ad(X) \circ ad(\beta^{-1}(Y)) \circ \beta^{-1}, \; X, Y \in \mathfrak{g},$$

and this, in turn, yields

$$\kappa_{\mathfrak{g}}(\beta(X), Y) = \kappa_{\mathfrak{g}}(X, \beta^{-1}(Y)).$$

Now, for any $X, Y \in \mathfrak{g}$, we have

$$\begin{aligned}
\kappa_{\mathfrak{g}}(\theta \circ \beta \circ \theta \circ \beta^*(X), Y) &= \overline{\kappa_{\mathfrak{g}}(\beta \circ \theta \circ \beta^*(X), \theta(Y))} \\
&= \overline{\kappa_{\mathfrak{g}}(\theta \circ \beta^*(X), \beta^{-1} \circ \theta(Y))} \\
&= \kappa_{\mathfrak{g}}(\beta^*(X), \theta \circ \beta^{-1} \circ \theta(Y)) \\
&= -B(\beta^*(X), \beta^{-1} \circ \theta(Y)) \\
&= -B(X, \theta(Y)) \\
&= \kappa_{\mathfrak{g}}(X, Y),
\end{aligned}$$

and hence $\theta \circ \beta \circ \theta \circ \beta^*(X) = X$, proving (4.2.7).

Returning to the proof of $\alpha^* \in Aut(\mathfrak{g})$, it is enough to show $\alpha^*([X, Y]) = [\alpha^*(X), \alpha^*(Y)]$ for all $X, Y \in \mathfrak{g}$. Let $X, Y, Z \in \mathfrak{g}$. Then using (i) and (ii) above, we have

$$\begin{aligned}
B([\alpha^*(X), \alpha^*(Y)], Z) &= -B(\alpha^*(Y), [\theta \circ \alpha^*(X), Z]) \\
&= -B(Y, \alpha([\theta \circ \alpha^*(X), Z])) \\
&= -B(Y, [\alpha \circ \theta \circ \alpha^*(X), \alpha(Z)]) \\
&= -B(Y, [\theta \circ (\theta \circ \alpha \circ \theta \circ \alpha^*)(X), \alpha(Z)]) \\
&= -B(Y, [\theta(X), \alpha(Z)]) \\
&= B([X, Y], \alpha(Z)) \\
&= B(\alpha^*([X, Y]), Z).
\end{aligned}$$

Hence $\alpha^*([X, Y]) = [\alpha^*(X), \alpha^*(Y)]$, proving $\alpha^* \in Aut(\mathfrak{g})$. ■

Theorem 4.15 *Let $\mathfrak{g}$ be a semisimple complex analytic Lie algebra. There exist an $\mathbb{R}$-linear subspace $\mathcal{S}$ of $\mathfrak{g}$ and a compact subgroup K' of $Aut_0(\mathfrak{g})$ such that the map*

$$(\alpha, X) \mapsto \alpha \circ (\exp \circ ad(X)) : K' \times \mathcal{S} \to Aut_0(\mathfrak{g})$$

is an isomorphism of real analytic manifolds.

Proof. Let $\mathfrak{g}_0$, θ, and B_θ be as in Proposition 4.13. Then $Aut(\mathfrak{g})$ (and hence $Aut_0(\mathfrak{g})$) is a self-adjoint algebraic subgroup of $GL(\mathfrak{g}, \mathbb{C})$ by Theorem 4.14. According to Theorem 1.18, the multiplication map

$$\psi : (Aut_0(\mathfrak{g}) \cap \mathrm{U}(\mathfrak{g})) \times (Aut_0(\mathfrak{g}) \cap \mathcal{P}(\mathfrak{g})) \to Aut_0(G)$$

is an isomorphism of real analytic manifolds, where $\mathcal{P}(\mathfrak{g})$ is the subset of $GL(\mathfrak{g}, \mathbb{C})$ that consists of all positive definite Hermitian elements. Let

$$K' = \{\alpha \in Aut_0(\mathfrak{g}) : \alpha(\mathfrak{g}_0) \subset \mathfrak{g}_0\}.$$

Then $K' = Aut_0(\mathfrak{g}) \cap \mathrm{U}(\mathfrak{g})$, and $\mathcal{L}(K') = ad_{\mathfrak{g}}(\mathfrak{g}_0)$. To see this, let $\alpha \in Aut_0(\mathfrak{g})$. Then $\alpha \in \mathrm{U}(\mathfrak{g})$ if and only if $\alpha^{-1} = \alpha^*$, or equivalently, $\theta \circ \alpha = \alpha \circ \theta$ (see the equation (4.2.7) in the proof of Theorem 4.14), and the latter condition holds if and only if $\alpha(\mathfrak{g}_0) \subset \mathfrak{g}_0$, proving $K' = Aut_0(\mathfrak{g}) \cap \mathrm{U}(\mathfrak{g})$. We next show $\mathcal{L}(K') = ad_{\mathfrak{g}}(\mathfrak{g}_0)$. Since $\mathfrak{g}$ is semisimple, we have $ad_{\mathfrak{g}}(\mathfrak{g}) = Der(\mathfrak{g}) = \mathcal{L}(Aut(G))$, and $\mathcal{L}(K')$ is a subalgebra of $ad_{\mathfrak{g}}(\mathfrak{g})$. Let $X \in \mathfrak{g}$ such that $ad_{\mathfrak{g}}(X) \in \mathcal{L}(K')$. We need to show $X \in \mathfrak{g}_0$. For any $t \in \mathbb{R}$ and $Y \in \mathfrak{g}_0$, we have $\exp(t\, ad(X)) \in K'$, and hence

$$[X, Y] = ad(X)(Y) = \lim_{t \to 0} \frac{1}{t}(\exp(t\, ad(X))(Y) - Y) \in \mathfrak{g}_0$$

follows. This shows $ad(X)(\mathfrak{g}_0) \subset \mathfrak{g}_0$. Now write $X = X_1 + \sqrt{-1}X_2$, where $X_1, X_2 \in \mathfrak{g}_0$. For $Z \in \mathfrak{g}_0$, $[X, Z] = [X_1, Z] + \sqrt{-1}[X_2, Z] \in \mathfrak{g}$, and we then have

$$\sqrt{-1}[X_2, Z] = [X, Z] - [X_1, Z] \in \mathfrak{g}_0 \cap \sqrt{-1}\mathfrak{g}_0 = (0),$$

which shows that X_2 is central in $\mathfrak{g}_0$. But $\mathfrak{g}_0$ is semisimple, and hence $X_2 = 0$, proving that $X \in \mathfrak{g}_0$.

Since $\mathfrak{g}_0 \cong ad_{\mathfrak{g}}(\mathfrak{g}_0)$ is a compact semisimple Lie algebra, the subgroup K' is a compact semisimple real analytic subgroup of $Aut(\mathfrak{g})$ by Theorem 4.10.

Now let $\mathcal{S} = \sqrt{-1}\mathfrak{g}_0$. Then

$$Aut_0(\mathfrak{g}) \cap \mathcal{P}(\mathfrak{g}) = \exp \circ\, ad(\mathcal{S}).$$

In fact, $Aut_0(\mathfrak{g}) \cap \mathcal{P}(\mathfrak{g})$ consists of $\exp(ad(X))$, where $X \in \mathfrak{g}$ and $ad(X)$ is Hermitian, i.e., $ad(X)$ satisfies the Hermitian condition:

$B([X,Y],Z) = B(Y,[X,Z])$. But this condition is equivalent to $\theta(X) = -X$, i.e., $X \in \sqrt{-1}\mathfrak{g}_0 = \mathcal{S}$. This may be seen as follows. For $Y, Z \in \mathfrak{g}$, the Hermitian condition yields

$$\begin{aligned} \kappa([X,Y],\theta(Z)) &= \kappa(Y,[\theta(X),\theta(Z)]) \\ &= -\kappa([\theta(X),Y],\theta(Z)) \\ &= \kappa([-\theta(X),Y],\theta(Z)), \end{aligned}$$

from which it follows that $ad(X)$ is Hermitian if and only if $\theta(X) = -X$. ∎

4.3 Semisimple Complex Analytic Groups

We now study complex semisimple analytic groups and their maximal compact subgroups.

Theorem 4.16 *Let G be a semisimple complex analytic group with Lie algebra $\mathfrak{g}$, and let K be a real analytic subgroup of G such that $\mathcal{L}(K)$ is a compact real form of $\mathfrak{g}$. Then*

(i) *K is semisimple, and is a maximal compact subgroup of G;*

(ii) *$Z(G) \subset K$.*

Proof. Let $\mathfrak{k} = \mathcal{L}(K)$. $\mathfrak{k}$ (and hence K) is semisimple by Proposition 4.12. Under the hypothesis, $\mathfrak{g} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{k}$, and K is a compact semisimple real analytic subgroup of G by Theorem 4.10. The kernel of the adjoint representation Ad of G is the center $Z(G)$, which is discrete, and we have $G/Z(G) \cong Ad(G) = Aut_0(\mathfrak{g})$. Let K' denote the analytic subgroup of $Ad(G) = Aut_0(\mathfrak{g})$ corresponding to the subalgebra $ad_{\mathfrak{g}}(\mathfrak{k})$. By Theorem 4.15, $\mathfrak{g}$ contains an $\mathbb{R}$-linear subspace $\mathcal{S}$ of $\mathfrak{g}$ such that the map

$$(\alpha, X) \mapsto \alpha \circ (\exp \circ ad(X)) : K' \times \mathcal{S} \to Ad(G) \qquad (4.3.1)$$

is an isomorphism of real analytic manifolds. Since $ad_{\mathfrak{g}}$ maps $\mathfrak{k}$ onto $\mathcal{L}(K') = ad_{\mathfrak{g}}(\mathfrak{k})$ isomorphically, $Ad(K) = K'$. Let $T = Ad^{-1}(K')$. Then T is a closed real Lie subgroup of G, and the map

$$\phi : G/T \to Ad(G)/K',$$

defined by $\phi(xT) = Ad(x)K'$ for $x \in G$, is a homeomorphism. $Ad(G)/K'$ is simply connected, as it is homeomorphic with the real Euclidean space $\mathcal{S}$ by (4.3.1), and hence G/T is simply connected. This implies that T is connected, and $T = K$ follows. In particular, we have $Z(G) \subset K$, proving (ii).

Now we prove (i). Suppose K is *not* maximal, and choose a compact maximal subgroup M of G that contains K. Then M is semisimple. To see this, it is enough to show that the center of M is finite. Let $x \in Z(M)$. Then $Ad(x)$ is trivial on $\mathfrak{k}$ and hence also on $\mathfrak{g}$, because $\mathfrak{g}$ is spanned (over $\mathbb{C}$) by $\mathfrak{k}$. This shows that $x \in Z(G)$, and $Z(M) \subset Z(G)$ follows. Since $Z(G)$ is finite, so is $Z(M)$, proving that M is semisimple. Now write $\mathcal{L}(M) = \mathfrak{k} \oplus \sqrt{-1}R$, where $R = \sqrt{-1}\mathcal{L}(M) \cap \mathfrak{k}$. Since M contains K properly, $R \neq \{0\}$. R is an ideal of $\mathfrak{k}$. In fact,

$$[R, \mathfrak{k}] \subset [\sqrt{-1}\mathcal{L}(M), \mathcal{L}(M)] = \sqrt{-1}\mathcal{L}(M)$$

and

$$[R, \mathfrak{k}] \subset [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k},$$

and these imply that $[R, \mathfrak{k}] \subset \sqrt{-1}\mathcal{L}(M) \cap \mathfrak{k} = R$, proving that R is an ideal of $\mathfrak{k}$. Then $R + \sqrt{-1}R$ is an ideal of $\mathfrak{k} + \sqrt{-1}\mathfrak{k} = \mathfrak{g}$. Let P denote the complex analytic subgroup of G whose Lie algebra is $R + \sqrt{-1}R$. Then P is compact. In fact, the ideal $\mathcal{L}(P)$ of the semisimple Lie algebra $\mathcal{L}(M)$ is again semisimple, and is therefore compact. This implies that P is compact. As a compact complex analytic group, P is abelian (Theorem 1.19), and hence $P = (1)$, proving $R = 0$. This shows that $K = M$, and K is a maximal compact subgroup of G. ∎

Corollary 4.17 *Let G be a semisimple complex analytic group. Then the center of G is finite.*

Proof. By Theorem 4.16, $Z(G) \subset Z(K)$. Since K is a compact semisimple analytic group, $Z(K)$ and hence $Z(G)$ is finite. ∎

Corollary 4.18 *A maximal semisimple complex analytic subgroup of a complex analytic group G is closed.*

Proof. Let S be a maximal semisimple complex analytic subgroup of G, and let R denote the radical of G so that we have $G = RS$ and $S \cap R$ is discrete. As a discrete subgroup of S, $S \cap R$ is a central subgroup of the complex semisimple group S, and therefore it is finite by Theorem 4.17. Now we form the (external) semidirect product $R \rtimes S$ with respect to the conjugation by the elements of S on R, and consider the multiplication morphism $\mu : R \rtimes S \to G$, given by $\mu(s, r) = sr$. Since $S \cap R$ is finite, so is $\ker(\mu)$, and thus the surjective morphism μ is a closed map. In particular, μ maps the closed subgroup $\{1\} \times S$ of $R \rtimes S$ onto S, proving that S is closed in G. ∎

We now prove that every semisimple complex analytic group has a faithful complex analytic representation. We begin with the following lemma.

Lemma 4.19 *Let G be a faithfully representable complex analytic group G. If P is a finite central subgroup of G, then G/P is faithfully representable.*

Proof. Choose a faithful complex analytic representation

$$\rho : G \to GL(V, \mathbb{C}).$$

The finite subgroup $\rho(P)$ is trivially Zariski closed in $GL(V, \mathbb{C})$, and if N denotes the normalizer of $\rho(P)$ in $GL(V, \mathbb{C})$, then N is a linear algebraic group, and $\rho(P)$ is a normal algebraic subgroup of N. The quotient group $N/\rho(P)$, being equipped with the structure of an affine algebraic group, may be viewed as a linear algebraic group. The canonical morphism $\tau : N \to N/\rho(P)$ then becomes a rational representation of N whose kernel is $\rho(P)$. The kernel of $\tau \circ \rho$ is exactly P, and $\tau \circ \rho$ hence induces a faithful complex analytic representation of G/P. ∎

Theorem 4.20 *A semisimple complex analytic group has a faithful complex analytic representation.*

Proof. Let G be a semisimple complex analytic group with Lie algebra $\mathfrak{g}$, and let $\sigma : \widetilde{G} \to G$ be the universal covering of G. $\ker(\sigma)$ is a discrete central subgroup of $\widetilde{G}$, and is finite by Theorem 4.17. Our assertion follows as soon as we have shown that $\widetilde{G}$ has a faithful complex analytic representation by Lemma 4.19. Thus replacing G by $\widetilde{G}$, if necessary, we may assume that G itself is simply connected. Let K be a compact subgroup of G such that $\mathcal{L}(K)$ is a compact real form. Then K is a maximal compact subgroup of G by Theorem 4.16. We identify $\mathfrak{g}$ with $\mathbb{C}\otimes_{\mathbb{R}}\mathcal{L}(K)$, so that the differential of the inclusion $K \to G$ becomes the canonical injection $\mathcal{L}(K) \to \mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}(K)$. The compact group K has a faithful real analytic representation, say $\rho : K \to GL(V, \mathbb{R})$. Since G is simply connected, there is a complex analytic representation

$$\sigma : G \to GL(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C})$$

whose differential is the complexification of the differential $d\rho$:

$$1_{\mathbb{C}} \otimes d\rho : \mathbb{C} \otimes_{\mathbb{R}} \mathcal{L}(K) \to \mathbb{C} \otimes \mathfrak{gl}(V, \mathbb{R}) = \mathfrak{gl}(\mathbb{C} \otimes_{\mathbb{R}} V, \mathbb{C}).$$

Since $\ker(Ad_G)$ is finite and central, we see that $\ker(Ad_G) \subset K$. On the other hand, it is clear that the restriction of σ to K is faithful, and hence the direct sum of σ and Ad_G

$$\sigma \oplus Ad_G : G \to GL((\mathbb{C} \otimes_{\mathbb{R}} V) \oplus \mathfrak{g}, \mathbb{C})$$

is faithful. ∎

Remark 4.21 Theorem 4.20 is not true in general for real groups. That is, there exists a semisimple *real* analytic group having a finite center which does not admit any faithful analytic representations. To construct such a group, let D be the kernel of the universal covering:

$$\sigma : \widetilde{SL}(2, \mathbb{R}) \to SL(2, \mathbb{R}).$$

Then $D \cong \mathbb{Z}$. Choose a subgroup D_1 of D of index 2, and let $G = \widetilde{SL}(2, \mathbb{R})/D_1$. Then G is a semisimple real analytic group with finite center. We show that G does not admit any faithful analytic representations. Assume the contrary, and suppose we have a faithful analytic representation of G. We may then assume that G is a linear group, say $G \subset GL(n, \mathbb{C})$ for some $n > 0$. The Lie algebra $\mathfrak{g}$ of G is a subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, and has the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g}$ in $\mathfrak{gl}(n, \mathbb{C})$. Let $G_{\mathbb{C}}$ denote the complex analytic subgroup of $GL(n, \mathbb{C})$ corresponding to the complex Lie subalgebra $\mathfrak{g}_{\mathbb{C}}$ in $\mathfrak{gl}(n, \mathbb{C})$. Since $SL(2, \mathbb{C})$ is simply connected, the isomorphism $\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{g}_{\mathbb{C}}$ is the differential of the universal covering of $G_{\mathbb{C}}$:

$$\phi : SL(n, \mathbb{C}) \to G_{\mathbb{C}}.$$

Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\sigma_0} & SL(2,\mathbb{R}) \\ \downarrow & & \downarrow \\ G_{\mathbb{C}} & \xleftarrow[\phi]{} & SL(2,\mathbb{C}) \end{array}$$

where σ_0 is a two-fold covering of $SL(2,\mathbb{R})$ that is induced by σ, and the vertical maps are the inclusions. The diagram is commutative since the corresponding Lie algebra diagram is so. But this is absurd: the left vertical map is injective while the composite of the remaining three maps is not. ∎

4.4 Reductive Complex Analytic Groups

A complex Lie group G is called *reductive* if G has a faithful complex analytic representation and if *each* complex analytic representation of G is semisimple. The following proposition shows that the second condition in this definition is equivalent to the fullness of a compact subgroup.

Proposition 4.22 *Suppose G is a complex analytic group. Every complex analytic representation of G is semisimple if and only if G has a full compact subgroup.*

Proof. Assume that G has a full compact subgroup, Q say. Given a complex analytic representation $\rho : G \to GL(V,\mathbb{C})$, let W be a G-stable $\mathbb{C}$-linear subspace of V. Choose a Hermitian inner product $<\ ,\ >$ on V, so that $\rho|_Q$ is a unitary representation of Q (Lemma 4.6). The orthogonal complement $W^\perp$ to W in V (with respect to $<\ ,\ >$) is also a Q-stable $\mathbb{C}$-linear subspace of V. Since Q is full in G, $W^\perp$ is G-stable, proving that ρ is semisimple.

Conversely, assume that every complex analytic representation of G is semisimple. In particular, the adjoint representation of G (and hence of $\mathfrak{g}$) is semisimple, i.e., Lie algebra $\mathfrak{g}$ of G is reductive. Thus $\mathfrak{g}$ is the direct sum of its center and the commutator subalgebra $\mathfrak{g}'$, which is necessarily semisimple (see Theorem A.22). We then obtain $G = ZG'$, where Z is the identity component of the center of G, and G' is the commutator subgroup of G. Let T be the maximal compact analytic subgroup of Z. By Proposition 4.1, $Z = U \times T^*$, where T^* is the smallest analytic subgroup of Z that contains T, and where U is a complex vector subgroup of Z. Then T^*G' is a complex analytic subgroup of G, and we have a direct product

$$G = U \times (T^*G'). \tag{4.4.1}$$

In fact, we have $G = UT^*G'$, and we need to show $U \cap T^*G' = (1)$. Let $z \in U \cap T^*G'$, and write $z = xy$, where $x \in T^*$ and $y \in G'$. Then $y = x^{-1}z \in Z \cap G'$. But $Z \cap G'$ is finite as a central subgroup of the semisimple complex analytic group G' (Theorem 4.17), and it is therefore contained in the maximum compact subgroup T of Z. Hence $y \in T$, and $z = xy \in T^* \cap U = (1)$, proving $z = 1$. Thus $U \cap T^*G' = (1)$, proving (4.4.1).

Now we show $G = T^*G'$ by showing $U = (1)$. If $U \neq (1)$, the complex vector group U has a faithful unipotent complex analytic representation (Example 2.6 (ii)), and the composite of this with the projection $G \to U$ is a nontrivial unipotent representation of G. This contradicts the assumption that every complex analytic representation of G is semisimple, and $U = (1)$ follows. We now have $G = T^*G'$. The closed semisimple complex analytic subgroup G' has a full compact subgroup, Q say, by Theorem 4.16, and TQ is a full compact subgroup of G. ∎

Corollary 4.23 *Let G be a reductive complex analytic group. The identity component of the center of G is reductive, and is isomorphic with a complex torus.*

Proof. Retaining the notation used in the proof of Proposition 4.22, the maximal compact subgroup T of Z is full in Z, and Z is faithfully representable as a subgroup of the reductive group G. The assertion therefore follows from Corollary 4.4. ∎

Proposition 4.22 together with the conjugacy of maximal compact subgroups of a real analytic group provides the existence and the conjugacy of maximal reductive subgroups.

Corollary 4.24 *A faithfully representable complex analytic group has a maximal reductive complex analytic subgroup, and any two such are conjugate to each other.*

Proof. Let G be a faithfully representable complex analytic group, and let Q be a maximal compact subgroup of G. Then Q is full in the smallest complex analytic subgroup Q^* that contains Q. Thus every complex analytic representation of Q^* is semisimple by Proposition 4.22. Since Q^* is faithfully representable, Q^* is reductive. We now show that Q^* is a maximal reductive subgroup by showing that if P is any reductive complex analytic subgroup of G, Q^* contains a conjugate of P. (This also shows the conjugacy of maximal reductive subgroups.) There is a full compact subgroup C of P by Proposition 4.22. By conjugacy of compact subgroups, there exists an element $x \in G$ such that $x^{-1}Cx \subset Q$, and hence $x^{-1}Px \subset Q^*$. ■

Later (see Theorem 4.48), we shall prove the conjugacy of maximal reductive subgroups *directly*, i.e., without using the conjugacy of compact subgroups.

Proposition 4.25 *If R is a compact analytic subgroup of a faithfully representable complex analytic group G, then* $\dim_{\mathbb{R}} R \leq \dim_{\mathbb{C}} G$.

Proof. Let R^* be the smallest complex analytic subgroup of G that contains R. Then R^* is reductive by Proposition 4.22. Our assertion follows as soon as we have shown that $\dim_{\mathbb{R}} R \leq \dim_{\mathbb{C}} R^*$. Replacing G with R^* if necessary, we may assume that G itself is reductive. Under this added assumption, the Lie algebra of G is reductive, and hence we may write $G = ZG'$, where Z is the identity component of the center of G and G' is semisimple. Since G' is semisimple, its center is finite, and hence the central subgroup $Z \cap G'$ of G' is finite. By Corollary 4.23, Z is isomorphic with $(\mathbb{C}^*)^d$, where $d = \dim_{\mathbb{C}} Z$, and hence the real dimension of the maximal compact subgroup of Z is equal to $\dim_{\mathbb{C}} Z$. In particular, we have

$$\dim_{\mathbb{R}}(R \cap Z) \leq \dim_{\mathbb{C}} Z. \tag{4.4.2}$$

On the other hand, $R/(R \cap Z) \cong RZ/Z \subset G/Z$. Since G/Z is semisimple, and since $R/(R \cap Z)$ is compact, we have

$$\dim_{\mathbb{R}}(R/(R \cap Z)) \leq \dim_{\mathbb{C}} G/Z. \tag{4.4.3}$$

Combining (4.4.2) and (4.4.3), we obtain $\dim_{\mathbb{R}} R \leq \dim_{\mathbb{C}} G$. ■

The converse of Proposition 4.25 is not valid in general, but we have a partial converse.

Theorem 4.26 *Let G be a complex analytic group and let Q be a maximal compact subgroup of G. Suppose that Q is full in G and that* $\dim_{\mathbb{R}} Q \leq \dim_{\mathbb{C}} G$. *Then G is faithfully representable. In particular, G is reductive.*

Proof. Since G has a full compact subgroup, every complex analytic representation of G is semisimple by Proposition 4.22. In particular, the Lie algebra of G is reductive, and hence $G = ZG'$ and $Z \cap G'$ is finite, where Z is the identity component of the center of G. Let T be the maximal compact subgroup of Z, and let P be a real analytic subgroup of the semisimple analytic subgroup G' such that $\mathcal{L}(P)$ is a compact real form of $\mathcal{L}(G')$. Then P is a maximal compact subgroup of G' by Theorem 4.16. Q contains a conjugate of the compact group TP, and we have

$$\begin{aligned}\dim_{\mathbb{R}} T + \dim_{\mathbb{R}} P &= \dim_{\mathbb{R}}(TP)\\ &\leq \dim_{\mathbb{C}} Q \leq \dim_{\mathbb{C}} G\\ &= \dim_{\mathbb{C}} Z + \dim_{\mathbb{C}} G'.\end{aligned}$$

Since $\dim_{\mathbb{R}} P = \dim_{\mathbb{C}} G'$, it follows that $\dim_{\mathbb{R}} T \leq \dim_{\mathbb{C}} Z$. On the other hand, we have $T = Q \cap Z$, and since the real Lie algebra $\mathcal{L}(Q)$ is reductive and spans (over $\mathbb{C}$) the Lie algebra $\mathfrak{g}$ of G, the center of $\mathcal{L}(Q)$ spans $\mathcal{L}(Z)$ over $\mathbb{C}$. Therefore T is full in Z, and we have $\dim_{\mathbb{R}} T \geq \dim_{\mathbb{C}} Z$. This shows $\dim_{\mathbb{R}} T = \dim_{\mathbb{C}} Z$, and by Corollary 4.4, Z is a complex torus. Consequently, Z has a faithful complex analytic representation. We already know that the semisimple group G' has a faithful complex analytic representation (Theorem 4.20). Thus $Z \times G'$ is faithfully representable. Since $G \cong (Z \times G')/D$ for some finite central subgroup D of $Z \times G'$, G is faithfully representable by Lemma 4.19. ■

Remark 4.27 The condition $\dim_{\mathbb{R}} Q \leq \dim_{\mathbb{C}} G$ in Theorem 4.26 becomes the actual equality under the *fullness* assumption of Q in G. In this case, Theorem 4.26 states that if Q is full in G and $\dim_{\mathbb{R}} Q = \dim_{\mathbb{C}} G$, then G is reductive. That this condition actually characterizes the reductivity will be established later in Theorem 4.31. ■

4.5 Compact Subgroups of Reductive Groups

The following lemma is a special case of Proposition 1.34.

Lemma 4.28 *For real analytic groups G and H, we have*

$$(G \times H)^+ = G^+ \times H^+$$

and the canonical map $\gamma_{G \times H} = \gamma_G \times \gamma_H$. ■

Theorem 4.29 *If G is a compact real analytic group, the canonical map $\gamma : G \to G^+$ is injective, $\gamma(G)$ is a maximal compact subgroup of G^+, and $\mathcal{L}(\gamma(G))$ is a real form of $\mathcal{L}(G^+)$. In particular, G^+ is faithfully representable and hence is reductive.*

Proof. Since a real compact Lie group admits a faithful analytic representation, γ is injective, and $\mathcal{L}(\gamma(G)) = \mathrm{Im}(d\gamma)$ is a real form of $\mathcal{L}(G^+)$ by Proposition 1.30. It remains to show that $\gamma(G)$ is a maximal compact subgroup of G^+. We first prove the assertion for the special case in which G is either abelian or semisimple. If G is abelian, then it is a (real) torus, and hence we identify G with the subgroup of $(\mathbb{C}^*)^n$, where $\dim G = n$. Then $G^+ = (\mathbb{C}^*)^n$, and the inclusion becomes the canonical map $\gamma : G \to G^+$. In this case, $\gamma(G)$ is the maximal compact subgroup of G^+. Now assume that G is semisimple. Since $\mathcal{L}(\gamma(G)) = d\gamma(\mathcal{L}(G))$ is a compact real form of $\mathcal{L}(G^+)$, $\gamma(G)$ is a maximal compact subgroup of $\mathcal{L}(G^+)$ by Theorem 4.16. Having proved the assertion for these special cases, we assume now G is an arbitrary compact real analytic group. The Lie algebra of G is reductive, and hence we have $G = ZG'$, where Z denotes the identity component of the center of G. The multiplication map

$$\mu : Z \times G' \to G$$

is a covering morphism with finite kernel, and the induced morphism

$$\mu^+ : (Z \times G')^+ \to G^+$$

is also a covering morphism with finite kernel by Proposition 1.33. Hence μ^+ maps a maximal compact subgroup of $(Z \times G')^+$ onto a maximal compact subgroup of G^+. Let i and j denote the inclusion maps of Z and G' into G, respectively, and let α and β denote the canonical injections of Z and G' into $Z \times G'$, respectively. Then from $\mu \circ \alpha = i,\ \mu \circ \beta = j$, we obtain $\mu^+ \circ \alpha^+ = i^+,\ \mu^+ \circ \beta^+ = j^+$.

Now we know that $\gamma_Z(Z)$ and $\gamma_{G'}(G')$ are maximal compact in Z^+ and G'^+, respectively, and $\gamma_Z(Z) \times \gamma_{G'}(G')$ is therefore a maximal compact subgroup of $Z^+ \times (G')^+$. Since $Z^+ \times (G')^+ \cong (Z \times G')^+$ (see Lemma 4.28), we see that $\alpha^+(\gamma_Z(Z))\beta^+(\gamma_{G'}(G'))$ is a maximal compact subgroup of $(Z \times G')^+$. From the commutative diagram

$$\begin{array}{ccccc} Z^+ & \xrightarrow{\alpha^+} & (Z \times G')^+ & \xleftarrow{\beta^+} & G'^+ \\ & {\scriptstyle i^+} \searrow & \downarrow {\scriptstyle \mu^+} & \swarrow {\scriptstyle j^+} & \\ & & G^+ & & \end{array}$$

we have $\mu^+(\alpha^+(\gamma_Z(Z))\beta^+(\gamma_{G'}(G'))) = \gamma(Z)\gamma(G') = \gamma(G)$, proving that $\gamma(G)$ is a maximal compact subgroup of G^+.

The last assertion of the theorem follows from $\dim_{\mathbb{R}} \gamma(G) = \dim_{\mathbb{C}}(G^+)$ and Theorem 4.26. ■

We now generalize Proposition 4.3 to *any* complex analytic groups.

Theorem 4.30 *Let G be a complex analytic group, and suppose G has a full maximal compact subgroup Q. If ρ is a complex analytic representation of G such that $\rho|_Q$ is faithful, then ρ is faithful.*

Proof. Let $\sigma = \rho|_Q$. Since Q is compact, σ may be viewed as a unitary representation. Hence the faithful representation σ and its dual σ° generate Rep(Q) (see, e.g., [4], Prop. 3, p. 190). Since Q is full in G, two complex analytic representations of G that coincide on Q are identical. Hence we may conclude that ρ and its dual ρ° generate Rep(G), the set of all complex analytic representations of G. Then $\ker(\rho) = \ker(\rho^\circ)$ is contained in the kernel of *every* complex analytic representation of G. In fact, the subset $\mathcal{E}$ of Rep(G) consisting of all representations ϕ such that $\ker(\rho) \subset \ker(\phi)$ is easily seen to be closed (see §2.1 for definition) and contains $\ker(\rho)$ and $\ker(\rho^\circ)$, and hence $\mathcal{E} = \text{Rep}(G)$ follows. Thus we shall show that $\ker(\rho)$ is trivial by showing that G has a faithful complex analytic representation. Since G contains a full compact subgroup, every complex analytic representation of G is semisimple by Proposition 4.22, and hence the Lie algebra $\mathfrak{g}$ of G is reductive. We may write $G = ZG'$, where Z is the identity component of the center of G. Since Q is a maximal compact subgroup of G, $Q \cap Z$ must coincide with the maximum compact subgroup of Z. In fact, we have the following general statement: If K is a maximal compact subgroup of an analytic group L, and if A is any closed central subgroup of L, then $A \cap L$ is the maximal compact group of A. This follows directly from the conjugacy of maximal compact groups of L. In particular, $Q \cap Z$ is connected, and its Lie algebra is the center of $\mathcal{L}(Q)$. On the other hand, $\mathcal{L}(Q)$ is a real reductive Lie algebra, and since $\mathcal{L}(Q)$ spans $\mathfrak{g}$ over $\mathbb{C}$, the center of $\mathcal{L}(Q)$ spans the center of $\mathfrak{g}$ over $\mathbb{C}$. This shows that $Q \cap Z$ is full in Z, and by Proposition 4.3, ρ is faithful on Z. The semisimple complex analytic group G/Z has a faithful complex analytic representation, and hence G has a complex analytic representation τ such that $\ker(\tau) = Z$. The direct sum of ρ and τ is then a faithful complex analytic representation of G. ■

Now we are ready to characterize reductive groups in terms of its maximal compact subgroups.

Theorem 4.31 *Let G be a complex analytic group. The following are equivalent.*

(i) G is reductive.

(ii) If Q is a maximal compact subgroup of G, the inclusion $Q \to G$ is a universal complexification of Q.

(iii) G contains a maximal compact subgroup Q, which is full in G and satisfies $\dim_{\mathbb{R}} Q = \dim_{\mathbb{C}} G$.

Proof. (i)⇒(ii): Assume that G is reductive. Then Q is also a full maximal compact subgroup of Q^+ (Theorem 4.29). On the other hand, since G is faithfully representable as a reductive group, we may identify G with a complex analytic subgroup of a full complex linear group, $GL(V, \mathbb{C})$ say. Then the inclusion $Q \to GL(V, \mathbb{C})$ extends to a complex analytic representation $\rho : Q^+ \to GL(V, \mathbb{C})$. Since Q is full in G, ρ maps Q^+ *onto* G, and ρ is an injection by Theorem 4.30, proving that $\rho : Q^+ \cong G$.

(ii)⇒(iii) follows from Theorem 4.29.

(iii)⇒(i) is proved in Theorem 4.26 (see Remark 4.27). ■

Corollary 4.32 *If G is a reductive complex analytic group, then the $\mathbb{C}$-algebra $R(G)$ is finitely generated.*

Proof. By Theorem 4.31, G is the universal complexification of any of its maximal compact subgroups, Q say, and hence $R(G) \cong R(Q)$ by Proposition 2.18. Since Q is a compact Lie group, the $\mathbb{C}$-algebra $R(Q)$ is known to be finitely generated (see, e.g., [4], Chap. VI), and $R(G)$ is thus finitely generated. ■

We next show that a group extension of a reductive group by a reductive group is again reductive. To that end, we first prove

Lemma 4.33 *Suppose that $\sigma : G_1 \to G_2$ is a covering morphism of complex analytic groups with finite* $\ker(\sigma)$*. Then G_1 is reductive if and only if G_2 is reductive.*

Proof. Let K_1 be a maximal compact subgroup of G_1, and let $K_2 = \sigma(K_1)$. Then K_2 is also a maximal compact subgroup of G_2, and $\sigma^{-1}(K_2) = K_1$. Since σ is a covering morphism, K_1 is full in G_1 if and only if K_2 is full in G_2, and we also have

$$\dim_{\mathbb{C}} G_1 = \dim_{\mathbb{C}} G_2 \ ; \dim_{\mathbb{R}} K_1 = \dim_{\mathbb{R}} K_2. \qquad (4.5.1)$$

By Theorem 4.31, G_i is reductive if and only if K_i is full in G_i, and $\dim_{\mathbb{R}}(K_i) = \dim_{\mathbb{C}}(G_i)$ for $i = 1, 2$. This, together with (4.5.1), implies that G_1 is reductive if and only if G_2 is reductive, proving our lemma. ■

The following lemma is an easy generalization of the well-known result of Iwasawa, which states that if a locally compact group G contains a closed normal real vector subgroup V such that G/V is compact, then G splits over V, i.e., G is a semidirect product of V and a compact subgroup.

Lemma 4.34 *Let G be a locally compact group which contains a simply connected solvable real analytic group S as a closed normal subgroup, and suppose that G/S is compact. Then G is a semidirect product of S and a compact subgroup.* ■

Theorem 4.35 *Let G be a complex analytic group, and K be a closed normal complex analytic subgroup of G. If K and G/K are reductive, then so is G.*

Proof. We first reduce to the case where K is abelian. Let Z be the identity component of the center of K. Then Z is reductive by Corollary 4.23. Now we show that G/Z is reductive. K/Z is a closed semisimple normal complex analytic subgroup of G/Z. In general, if $\mathfrak{b}$ is a semisimple ideal of a finite dimensional Lie algebra $\mathfrak{h}$ such that $\mathfrak{h}/\mathfrak{b}$ is reductive, then $\mathfrak{b}$ is a direct summand of $\mathfrak{h}$. Applying this to the Lie algebra of G/Z, we can find a normal complex analytic subgroup U of G/Z such that $G/Z = (K/Z)U$ and that $(K/Z) \cap U$ is a finite central subgroup of K/Z. Thus the multiplication $(x, u) \mapsto xu$ defines a covering morphism

$$(K/Z) \times U \longrightarrow G/Z$$

with finite kernel. Let $\sigma : U \to G/K$ be the restriction to U of the canonical morphism $G/Z \to G/K$. σ is a covering morphism, and its kernel is the finite group $(K/Z) \cap U$. By Lemma 4.33, U is a reductive complex analytic group. Since the semisimple group K/Z is reductive, the product $(K/Z) \times U$ is reductive, and it follows again from Lemma 4.33 that G/Z is reductive. Replacing K by its central subgroup Z, if necessary, we may assume that K itself is abelian. In that case, the maximal compact subgroup T of the abelian group K is normal (and hence central) in G, and the kernel K/T of the canonical morphism $\pi : G/T \to G/K$ is a real vector group. Let Q be a maximal compact subgroup of G/K. Then $\pi^{-1}(Q)/(K/T) \cong Q$, and $\pi^{-1}(Q)$ splits over the vector subgroup K/T by Lemma 4.34. If P/T is a compact subgroup of $\pi^{-1}(Q)$ such that

$$\pi^{-1}(Q) = (K/T) \cdot (P/T)$$

(semidirect product), then P is easily seen to be a maximal compact subgroup of G, and π maps P/T isomorphically onto Q. It follows that

$$\dim_{\mathbb{R}} P - \dim_{\mathbb{R}} T = \dim_{\mathbb{R}}(P/T) = \dim_{\mathbb{R}} Q.$$

On the other hand, since K and G/K are reductive, we have from Theorem 4.31 that

$$\dim_{\mathbb{R}} Q = \dim_{\mathbb{C}}(G/K) = \dim_{\mathbb{C}}(G) - \dim_{\mathbb{C}}(K); \dim_{\mathbb{R}} T = \dim_{\mathbb{C}} K.$$

It follows from the equalities above that $\dim_{\mathbb{R}} P = \dim_{\mathbb{C}} G$. Finally, T and Q are full in K and Q, respectively, and hence the maximal compact subgroup P is full in G. Now the reductivity of G follows from Theorem 4.31. ■

In §5.5, we shall also establish that if any two of the groups G, K, and G/K in Theorem 4.35 are reductive, then so is the remaining group.

4.6 Representation Radical

For a complex analytic group G, the intersection $N(G)$ of all kernels of semisimple complex analytic representations of G is called the *representation radical* of G. This is a closed normal complex Lie subgroup of G satisfying the property that the restriction to $N(G)$ of *every* complex analytic representation of G is unipotent, and it may be characterized as the largest subgroup of G with this property. Similarly, the representation radical $N(G)$ of a *real* analytic group G is defined to be the intersection of all real analytic representations of G. It follows immediately from the definition of $R(G, N(G))$ (see §2.6) that $R(G) = R(G, N(G))$.

In this section, we shall determine the representation radical of a faithfully representable complex analytic group.

Proposition 4.36 *Let G be a complex analytic group, and let N be the radical of the commutator subgroup G' of G. If $\rho : G \to GL(V, \mathbb{C})$ is a complex analytic representation, $\rho(N)$ is a unipotent subgroup of $GL(V, \mathbb{C})$, and hence it is closed and simply connected.*

Proof. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{r}$ (resp. $\mathfrak{n}$) denote the radical of $\mathfrak{g}$ (resp. $[\mathfrak{g}, \mathfrak{g}]$) so that $\mathcal{L}(N) = \mathfrak{n}$. By Proposition A.18, $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$, and the differential $d\rho$ maps $\mathfrak{n}$ onto a Lie subalgebra of $\mathfrak{gl}(V, \mathbb{C})$ consisting of nilpotent linear transformations of V. The complex analytic subgroup $\rho(N)$ is therefore unipotent, and by Corollary 3.13, $\rho(N)$ is closed and simply connected. ■

Proposition 4.37 *For a faithfully representable complex analytic group G, the commutator subgroup G' of G is closed in G.*

Proof. Let ρ be a faithful complex analytic representation of G. Let R be the radical of G and let S be a maximal semisimple analytic subgroup of G. We form the (external) semidirect product $R \rtimes S$

with respect to the homomorphism $S \to Aut(R)$ induced by the conjugation, and we consider the multiplication map $\mu : R \rtimes S \to G$. This is a surjective homomorphism with kernel

$$\{(x, x^{-1}) : x \in R \cap S\}.$$

Since $R \cap S$ is finite as a central subgroup of S (Theorem 4.17), $\ker(\mu)$ is finite, and hence μ is a closed map. Moreover, μ maps the commutator subgroup $(R \rtimes S)'$ of $R \rtimes S$ onto G'. Therefore to show that G' is closed in G, it is enough to show that $(R \rtimes S)'$ is closed in $R \rtimes S$. Let $\mathfrak{g}$ denote the Lie algebra of G, and let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. We first note that $(R \rtimes S)' = [G, R] \times S$, where $[G, R]$ is the analytic subgroup corresponding to the subalgebra $[\mathfrak{g}, \mathfrak{r}]$. Let N be the radical of the commutator subgroup G'. In the proof of Proposition 4.36, we have established $[G, R] = N$, and hence by Proposition 4.36 $\rho([G, R])$ is a closed unipotent subgroup of the full linear group of the representation space of ρ. It follows that $\rho([G, R])$ is closed in $\rho(R)$, and, consequently, $[G, R]$ is closed in R. It is now clear that $(R \rtimes S)' = [G, R] \times S$ is closed in $R \rtimes S$. ∎

Theorem 4.38 *Let G be a faithfully representable complex analytic group, and let N be the radical of the commutator subgroup G' of G. Then*

(i) *N is closed, nilpotent, and simply connected;*

(ii) *If ρ is a complex analytic representation of G, then $\rho(N)$ is unipotent;*

(iii) *G/N has a faithful semisimple complex analytic representation.*

Proof. (ii) is proved in Proposition 4.36.

We fix a faithful complex analytic representation $\sigma : G \to GL(V, \mathbb{C})$ of G. By Proposition 4.36, the subgroup $\sigma(N)$ is closed in $GL(V, \mathbb{C})$ and is simply connected, and this readily implies that $N = \sigma^{-1}(\sigma(N))$ is closed in G and is simply connected, proving (i).

We now prove (iii). Our proof is divided into several parts.

(a) G/G' admits a faithful complex analytic representation.

Let R denote the radical of G, and let S be a maximal semisimple complex analytic subgroup of G. $R \cap S$ is finite as a central subgroup of the semisimple complex analytic group S. The relations

$$G = RS,\ G' = NS,\ R \cap G' = R \cap NS = N(R \cap S)$$

imply that

$$(R \cap G')/N = N(R \cap S)/N \cong (R \cap S)/(N \cap S)$$

is finite, and the isomorphisms

$$G/G' = RG'/G' \cong R/(R \cap G') \cong (R/N)/((R \cap G')/N)$$

show that G/G' is the quotient group of R/N by the finite subgroup $(R \cap G')/N$. By Lemma 4.19, it is therefore enough to show that R/N admits faithful complex analytic representation. Let T be the maximum compact subgroup of the abelian group R/N, and let $\pi : R \to R/N$ be the canonical map. Since $\pi^{-1}(T)/N = T$ is compact and since N is a simply connected nilpotent subgroup, $\pi^{-1}(T)$ contains a compact subgroup P such that $\pi^{-1}(T) = PN$ and $P \cap N = (1)$ (Lemma 4.34). Let σ' be the semisimple representation associated with σ (see §2.1). Since P is compact, $\sigma|_P$ is semisimple, and hence $\sigma|_P$ and $\sigma'|_P$ are equivalent (Lemma 2.2). This implies that $\sigma' : G \to GL(V', \mathbb{C})$ is faithful on P. On the other hand, σ' maps N to 1, and hence it induces a representation τ of G/N which is faithful on $T = PN/N$. By Proposition 4.3, τ is faithful on the smallest analytic subgroup T^* of R/N that contains T. Write $R/N = T^* \times U$, where U is a complex vector subgroup (Proposition 4.1). Since U is faithfully representable, it follows that R/N admits a faithful complex analytic representation.

(b) The abelian group G/G' has a faithful semisimple complex analytic representation.

Write $G/G' = C^* \times W$, where C^* is the smallest complex analytic subgroup of G that contains the maximum compact subgroup C of G/G', and W is a complex vector subgroup of G/G' (Lemma 4.1). Since G/G' is faithfully representable by (a), so is C^*, and hence it is reductive by Proposition 4.22. On the other hand, the vector group W also has a faithful semisimple complex analytic representation (see Example 2.6). Thus $G/G' = C^* \times W$ has a faithful semisimple complex analytic representation.

(c) G/N has a semisimple complex analytic representation ϕ whose kernel is G'/N.

Since G' is closed in G (Proposition 4.37) and contains N, $(G/N)'$ is also closed in G/N, and G/G' is isomorphic to the quotient group $(G/N)/(G/N)'$. G/G' has a faithful semisimple complex analytic representation by (b), and the composite of this with the canonical map $G/N \to (G/N)/(G/N)'$ provides a semisimple complex analytic representation ϕ of G/N whose kernel is $(G/N)' = G'/N$.

(d) The quotient group G/N has a faithful semisimple complex analytic representation.

Let S be a maximal semisimple complex analytic subgroup of G, and let σ' denote the semisimple representation associated with σ (see §2). Then the representations $\sigma|_S$ and $\sigma'|_S$ are equivalent by Lemma 2.2. In particular, σ' is faithful on S. Since $\sigma'(N)$ is trivial, σ' induces a representation ψ of G/N, which is faithful on $SN/N = G'/N = (G/N)'$. The direct sum of ϕ and ψ is a faithful semisimple complex analytic representation of G/N. ■

Corollary 4.39 *Let G be a faithfully representable complex analytic group. Then the representation radical $N(G)$ of G coincides with the radical of the commutator subgroup G' of G.*

Proof. Let N denote the radical of the commutator subgroup G'. The composite of the canonical morphism $G \to G/N$ and the faithful semisimple representation of G/N in Theorem 4.38 (iii) provides a semisimple representation of G with kernel N, and hence $N(G) \subset N$. On the other hand, $N \subset N(G)$ by Theorem 4.38 (ii), and $N = N(G)$ follows. ■

The following proposition establishes the relationship between $N(G)^+$ and $N(G^+)$, when G is a real group.

Proposition 4.40 *Let G be a faithfully representable real analytic group. Then $N(G)^+ \cong N(G^+)$.*

Proof. Put $N = N(G)$ and $N^+ = N(G)^+$, and let $\mathfrak{g}$, $\mathfrak{g}^+$, $\mathfrak{n}$, and $\mathfrak{n}^+$ denote the Lie algebras of G, G^+, N, and N^+, respectively. A real analytic representation $\rho : G \to GL(V, \mathbb{C})$ is semisimple if and only if the induced representation $\rho^+ : G^+ \to GL(V, \mathbb{C})$ is semisimple, and hence it follows from the definition of the representation radical that the canonical injection $\gamma : G \to G^+$ maps N into $N(G^+)$. Let γ_0 denote the restriction of γ to N. Thus there is a unique complex analytic homomorphism

$$\gamma' : N^+ \to N(G^+)$$

such that $\gamma' \circ \eta = \gamma_0$, where $\eta : N \to N^+$ denotes the canonical injection. We claim that γ' is an isomorphism. We first note

$$\dim_{\mathbb{C}} N(G^+) = \dim_{\mathbb{C}} N^+ (= \dim_{\mathbb{R}} N).$$

In fact, if we identify $\mathfrak{g}$ with its image $d\gamma(\mathfrak{g})$ in $\mathfrak{g}^+$ for convenience, then $\mathfrak{g}$ is a real form of the complex Lie algebra $\mathfrak{g}^+$, and the identity above follows from

$$\begin{aligned} \mathcal{L}(N(G^+)) &= Rad([\mathfrak{g}^+, \mathfrak{g}^+]) \\ &= Rad(\mathbb{C} \otimes [\mathfrak{g}, \mathfrak{g}]) \\ &= \mathbb{C} \otimes Rad([\mathfrak{g}, \mathfrak{g}]) \\ &= \mathbb{C} \otimes \mathfrak{n}. \end{aligned}$$

This equality shows that $d\gamma(\mathfrak{n})$ spans $\mathcal{L}(N(G^+))$ over $\mathbb{C}$. Since $d\eta(\mathfrak{n})$ is a real form of $\mathfrak{n}^+$, it follows from $d\gamma' \circ d\eta = d\gamma_0$ that $d\gamma'$ is surjective, and hence is an isomorphism because of $\dim_{\mathbb{C}} N(G^+) = \dim_{\mathbb{C}} N^+$. This means that γ' is a covering morphism of $N(G^+)$. But $N(G^+)$ is simply connected, and so γ' must be an isomorphism, proving our assertion. ∎

4.7 Faithfully Representable Groups

The purpose of this section is to study complex analytic groups which admit faithful analytic representations. We begin with a result which is analogous to the Iwasawa's Theorem (cf., Lemma 4.34) for analytic groups, where compact groups play the role of reductive groups.

Proposition 4.41 *Let G be a complex analytic group, and let S be a closed simply connected normal solvable complex analytic subgroup of G. If G/S is reductive, then G is a semidirect product of S and a maximal reductive subgroup of G.*

Proof. Choose a maximal compact subgroup Q of the reductive group G/S, and let K be the inverse image of Q in G under the canonical morphism $\pi : G \to G/S$. Thus K/S is compact, and we have a semidirect product $K = P \cdot S$ of real Lie groups (Lemma 4.34), where P is a compact subgroup of K, which is necessarily a maximal compact subgroup of G. As a reductive group, G/S has a faithful complex analytic representation, and hence there exists a complex analytic representation ρ of G whose kernel is exactly S. The representation ρ is faithful on P because P meets S trivially, and hence by Theorem 4.30, ρ is faithful on the smallest complex analytic subgroup P^* that contains P. Thus $P^* \cap S = (1)$. Since $Q = \pi(P) \subset \pi(P^*)$, $\pi(P^*) = P^*S/S = G/S$ follows from the fullness of Q in G/S. Thus $G = P^*S$, proving that G is a semidirect product of S and the reductive subgroup P^*. Since P is a maximal compact subgroup of G, P^* is a maximal reductive subgroup of G, and this completes the proof. ∎

Before we discuss the decomposition theorem (Theorem 4.43), we first consider the groups admitting a faithful *semisimple* complex analytic representation.

Proposition 4.42 *Let G be a complex analytic group, and suppose that G has a faithful semisimple complex analytic representation. Then $G = U \times H$, where U is a complex vector group, and H is a reductive complex analytic group.*

Proof. Under the hypothesis, the Lie algebra $\mathfrak{g}$ of G admits a semisimple faithful representation, and therefore $\mathfrak{g}$ is reductive by Theorem A.22. We have $G = ZG'$, where Z denotes the identity component of the center of G. If T is the maximum compact subgroup of the abelian group Z, then $Z = U \times T^*$, where U is a complex vector subgroup of Z and T^* denotes the smallest complex analytic subgroup that contains T. If we put $H = T^*G'$, then $G = U \times H$ (see (4.4.1) in the proof of Proposition 4.22). It remains to show that H is reductive. Since H contains a compact full subgroup, namely, TQ, where Q is a maximal compact subgroup of the semisimple group G', H is reductive by Proposition 4.22. ∎

The Decomposition Theorem The result below characterizes the faithfully representable complex analytic groups.

Theorem 4.43 *Let G be a complex analytic group. G is faithfully representable if and only if G is a semidirect product $H \cdot K$ where K is a closed simply connected normal solvable complex analytic subgroup, and H is a reductive analytic subgroup of G.*

Proof. First assume that G is faithfully representable, and let N denote the representation radical of G. Since G/N has a faithful semisimple complex analytic representation (Theorem 4.38, (iii)), $G/N = U \times R$, where R is a reductive complex analytic subgroup and U is a complex vector subgroup of G/N (Proposition 4.42). Let K be the subgroup of G containing N such that $K/N = U$. K is a simply connected closed normal complex analytic subgroup of G, and $G/K \cong R$ is reductive. By Proposition 4.41, G is a semidirect product of K and a reductive group H.

Now we assume $G = H \cdot K$ (semidirect product), where K is a closed simply connected solvable normal complex analytic subgroup of G, and H is a reductive complex analytic subgroup. We claim that G is faithfully representable. In fact, we prove this in somewhat stronger form for later use: If M_0 denotes the maximum nilpotent normal complex analytic subgroup of K, then G admits a faithful M_0-unipotent complex analytic representation. Let $\mathfrak{g} = \mathcal{L}(G)$, and put $\mathfrak{k} = \mathcal{L}(K)$, $\mathfrak{h} = \mathcal{L}(H)$, and $\mathcal{L}(M_0) = \mathfrak{m}_{\mathfrak{o}}$. By Proposition 3.15, there is a faithful M_0-unipotent complex analytic representation ϕ_0 of K, and $\dim U(K, M_0) = \dim_{\mathbb{C}} K < \infty$. We also have $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{m}_{\mathfrak{o}}$ (by Proposition A.20), and this implies that the canonical action of H on K/M_0 induced by conjugation is trivial. By Theorem 3.7, the representation ϕ_0 extends to an M_0-unipotent complex analytic representation ϕ of G. On the other hand, the composite of the projection $G \to H$ and any faithful complex analytic representation of the reductive group H defines a representation σ of G having K as its kernel. The direct sum of ϕ and σ is a faithful M_0-unipotent complex analytic representation of G. ■

Nucleus of Lie Groups By a *nucleus* of a complex Lie group G, we mean a closed, normal, solvable, and simply connected complex analytic subgroup K of G such that G/K is reductive. We note that if K is a nucleus of a complex analytic group G, then G is a semidirect product of K and a maximal reductive analytic subgroup of G by Theorem 4.41. Therefore by Theorem 4.43, we see that G admits a nucleus if and only if G is faithfully representable. We also have

Proposition 4.44 *A nucleus of a faithfully representable complex analytic group G contains the representation radical of G.*

Proof. Let K be a nucleus of G. Then G/K is reductive, and hence has a faithful semisimple complex analytic representation ϕ.

The composite of this representation with the canonical morphism $G \to G/K$ yields a semisimple complex analytic representation of G with kernel K. From the definition of $N(G)$, we see that $N(G) \subset K$. ∎

Now we generalize Proposition 3.15 to any faithfully representable groups. For that we need

Lemma 4.45 *Let G be a faithfully representable complex analytic group, and let M be the maximal nilpotent normal complex analytic subgroup of G. If M is simply connected, then there is a nucleus K of G such that $M \subset K$.*

Proof. Let R denote the radical of G. Then $M \subset R$, and the abelian complex analytic group R/M is a direct product of a complex vector subgroup and a complex torus. Since M is assumed to be simply connected, R contains a simply connected solvable normal complex analytic subgroup K such that $M \subset K$ and that R/K is a complex torus. Since G/R is semisimple, and since $G/R \cong (G/K)/(R/K)$, it follows from Theorem 4.35 that G/K is reductive, i.e., K is a nucleus of G. ∎

Theorem 4.46 *Let G be a faithfully representable complex analytic group, and let M be the maximum nilpotent normal complex analytic subgroup of G. If M is simply connected, then there is a faithful M-unipotent complex analytic representation of G.*

Proof. Let K be a nucleus of G that contains M (Lemma 4.45). Clearly M is contained in the maximum nilpotent normal complex analytic subgroup of K, and hence, by the last part of the proof of Theorem 4.43, G admits a faithful M-unipotent complex analytic representation. ∎

4.8 Conjugacy of Reductive Subgroups

In this section we reestablish the conjugacy of maximal reductive subgroups directly, i.e., without reference to the conjugacy of maximal compact subgroups (cf., Corollary 4.24).

Let G be an abstract group, and, for a G-module V, a function $f : G \to V$ is called a 1-*cocycle* of G (with values) in V if f satisfies

$$f(xy) = f(x) + x \cdot f(y) \text{ for } x, y \in G.$$

A cocycle f is called a 1-*coboundary* if there exists an element $v \in V$ such that

$$f(x) = v - x \cdot v \text{ for all } x \in G.$$

In the case where G is a Lie group and V is an analytic G-module, we can talk about *analytic* cocycles or coboundaires of G. Throughout the remainder of this section, $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$, and also the analyticity refers to either a real or complex one depending on whether G is a real or complex analytic group. The following is a key lemma.

Lemma 4.47 *Let G be a reductive analytic group. For any finite dimensional analytic G-module V, every analytic 1-cocycle of G in V is a coboundary.*

Proof. Let $f : G \to V$ be a 1-cocycle, and we define an action of G on the direct sum $V \oplus \mathbb{K}$ by

$$x \cdot (v, a) = (af(x) + x \cdot v, a),$$

where $x \in G$, $v \in V$, and $a \in \mathbb{K}$. The cocycle identity implies that the above defines an analytic G-module structure on $V \oplus \mathbb{K}$, which contains V as a G-submodule. Since G is reductive, there is a G-module complement for V in $V \oplus \mathbb{K}$, which is 1-dimensional. There is an element $v \in V$ such that $(v, 1)$ spans this complement over $\mathbb{K}$. Since the reductive group G acts trivially on any 1-dimensional G-module, we have $f(x) + x \cdot v = v$ for all $x \in G$, proving that f is a coboundary. ∎

Theorem 4.48 *Let G be a faithfully representable analytic group, and suppose G is a semidirect product $G = K \cdot P$, where K is a nucleus of G and P is a closed reductive analytic subgroup of G. If Q is any closed reductive Lie subgroup of G, then there exists an element $u \in N(G)$ such that $uQu^{-1} \subset P$. In particular, P is a maximal reductive subgroup, and any two maximal reductive subgroups are conjugate by an element of $N(G)$.*

Proof. We put $N = N(G)$. Since N is the radical of the commutator subgroup $[G, G]$ (Corollary 4.39), we have $N = [G, R]$ by Proposition A.18, where R is the radical of G. We prove the assertion using induction on $\dim_{\mathbb{K}} N$. First we assume $N = (1)$. Then R is central in G, and since $K \subset R$, we see that the semidirect product $G = K \cdot P$ is a direct product. Let η denote the restriction to Q of the projection map from $G = K \cdot P$ onto K. Then $\eta(Q)$ is a reductive subgroup of the vector group K, and hence must be trivial. This shows $Q \subset P$, proving our assertion in the case of $N = (1)$. Now we assume that $N \neq (1)$. The center Z of N is a normal (vector) subgroup of G, and if $\pi : G \to G/Z$ denotes the canonical morphism, then by induction hypothesis there exists an element $u' \in N(G/Z) = N(G)/Z$ such that $u'\pi(Q)(u')^{-1} \subset \pi(P)$. This shows that $uQu^{-1} \subset ZP$ for some $u \in N(G)$, and replacing uQu^{-1} by Q, if necessary, we may assume that $Q \subset ZP$. Noting that the subgroup ZP is a semidirect product (i.e., $Z \cap P = (1)$), let $\zeta : Q \to Z$ and $\pi : Q \to P$ denote the projections (restricted to Q) of ZP onto Z and P, respectively, so that $x = \zeta(x)\pi(x)$. We write the group operation of the vector group Z additively, and view it as an analytic Q-module, where Q acts on Z by $x \cdot z = \pi(x)z\pi(x)^{-1}$. Then $\zeta(xy) = \zeta(x) + x \cdot \zeta(y)$ for all $x, y \in Q$, i.e., ζ is a cocycle of Q with values in Z. By the lemma above, it is a coboundary, and hence there exists an element $v \in Z$ such that

$$\zeta(x) = v - x \cdot v$$

for all $x \in Q$. Thus, for $x \in Q$, we have

$$x\pi(x)^{-1} = \zeta(x) = v\pi(x)v^{-1}\pi(x)^{-1},$$

and this implies $v^{-1}xv = \pi(x) \in P$, proving $v^{-1}Qv \subset P$. ■

4.9 Unipotent Hull of Complex Lie Groups

Here we prove that the unipotent hull of a faithfully representable complex analytic group is finite-dimensional. See Remark 3.8 for its significance.

Proposition 4.49 *Let K be any nucleus of a faithfully representable complex analytic group G. Every $N(G)$-unipotent complex analytic representation of K extends to a complex analytic representation of G. In particular, the restriction map*

$$R(G) = R(G, N(G)) \to R(K, N(G))$$

is surjective.

Proof. Put $N = N(G)$. G may be written as $G = H \cdot K$ (semidirect product), where H is a maximal reductive complex analytic subgroup of G (Theorem 4.43). Then we have $[G, R] = N$, where R denotes the radical of G (see the proof of Proposition 4.36). $K \subset R$ yields $[G, K] \subset N$, and this implies that the action of H on K/N induced by conjugation action of H on G is trivial. We have $\dim U(K, N) < \infty$ by Proposition 3.15, and thus every N-unipotent complex analytic representation of K extends to a complex analytic representation of G by Theorem 3.7. The second assertion of the proposition follows from Lemma 3.5. ∎

Theorem 4.50 *If G is a faithfully representable complex analytic group, then*

$$\dim U(G) = \dim_{\mathbb{C}} K$$

where K is any nucleus of G.

Proof. Let $N = N(G)$. We have $R(G, N) = R(G)$; $U(G) = U(G, N)$, and the restriction morphism

$$R(G, N) \to R(K, N)$$

is a surjection by Proposition 4.49, and we have the exact sequence

$$1 \to U(K, N) \to U(G, N) \to U(H) \to 1 \qquad (4.9.1)$$

by Corollary 3.9. Since H is reductive, $U(H) = (1)$ (see §2.7), and the exactness of the sequence (4.9.1) gives an isomorphism

$$U(K, N) \cong U(G, N).$$

On the other hand, $\dim U(K, N) = \dim_{\mathbb{C}} K$ by Proposition 3.15, and hence we have $\dim U(G) = \dim_{\mathbf{C}} K$. ∎

Chapter 5

Algebraic Subgroups in Lie Groups

In this section we shall examine the question of when a complex Lie group G may be given the structure of an affine algebraic group which is compatible with its analytic group structure in the sense that the rational representations of G are exactly the complex analytic representations of G. To answer this question in a more general context, we introduce and study the notion of algebraic subgroups in a complex Lie group ([12], [23], [24]).

5.1 Affine Algebraic Structure in Lie Groups

A complex Lie subgroup K of a faithfully representable complex analytic group G is called an *algebraic subgroup* of G if the restriction algebra $R(G)_K$ (i.e., the restriction to K of the functions in $R(G)$) is finitely generated as a subalgebra of $R(K)$, and if K is an affine algebraic group with $R(G)_K$ as its polynomial algebra, i.e., $P(K) = R(G)_K$. Recalling from §2.3 that $Aut(R(G)_K)$ denotes the pro-affine algebraic group consisting of all proper automorphisms of the fully stable subalgebra $R(G)_K$, it follows immediately from the definition that a closed complex Lie subgroup K of G is an algebraic subgroup if and only if the following two conditions are satisfied:

(i) $R(G)_K$ is finitely generated;

(ii) The canonical map $x \mapsto \tau_x : K \to Aut_K(R(G)_K)$, where $\tau_x(f) = x \cdot f,\ f \in R(G)_K$, is an isomorphism of groups.

It is evident from the definition that, for any algebraic subgroup K of G, the affine algebraic group structure of K is determined *not* by the analytic group structure of K itself, but entirely by the analytic structure of the ambient group G.

Proposition 5.1 *Let K be a closed complex Lie subgroup K of a faithfully representable complex analytic group G. Then the following are equivalent.*

(i) *K is an algebraic subgroup of G.*

(ii) *The canonical injection $\tau : G \to A(G)$ maps K to an (affine) algebraic subgroup of the pro-affine algebraic group $A(G)$.*

If K is an algebraic subgroup of G, then the restriction to K of every complex analytic representation of G is rational.

Proof. (i) $\Rightarrow$(ii): We need to show that if $g \in P(A(G)) = R(G)$, then $g \circ \tau_K \in P(K)$. Write g as $g = \varepsilon \circ \psi$, where

$$\psi : A(G) \to GL(V, \mathbb{C})$$

is a rational representation, and $\varepsilon : End(V) \to \mathbb{C}$ is a linear function. Then $\psi \circ \tau : G \to GL(V, \mathbb{C})$ is a complex analytic representation, and hence $\varepsilon \circ (\psi \circ \tau) \in R(G)$. Thus

$$g \circ \tau_K = (g \circ \tau)_K = (\varepsilon \circ \psi \circ \tau)_K \in R(G)_K = P(K)$$

follows.

(ii)$\Rightarrow$(i): Since $\tau(K)$ is an (affine) algebraic subgroup of $A(G)$, $P(\tau(K))$ is finitely generated and we have $P(\tau(K)) = P(A(G))_{\tau(K)} = R(G)_{\tau(K)}$. Transferring the affine algebraic group structure of $\tau(K)$ to K by means of the group isomorphism τ_K, we obtain an affine algebraic group structure on K so that $P(K) = R(G)_K$.

For the last assertion, let $\rho : G \to GL(V, \mathbb{C})$ be a complex analytic representation. We must show that if η is any $\mathbb{C}$-linear function on the linear space $End_{\mathbb{C}}(V)$, then $\eta \circ \rho_K \in P(K)$. But $\eta \circ \rho \in R(G)$, and hence $\eta \circ \rho_K = (\eta \circ \rho)_K \in R(G)_K = P(K)$. ■

Note that an algebraic subgroup K of a faithfully representable complex analytic group G is necessarily closed. To see this, choose any faithful complex analytic representation $\rho : G \to GL(V, \mathbb{C})$. Then ρ_K is rational by Proposition 5.1, and $\rho(K)$ is therefore Zariski closed (and hence Euclidean closed) in $GL(V,\mathbb{C})$, and this implies that K is closed in G.

Remark 5.2 It follows from Proposition 5.1 that if A and B are algebraic subgroups of G such that A normalizes B, then AB is an algebraic subgroup of G. In fact, $\tau(A)$ and $\tau(B)$ are algebraic subgroups of the pro-affine algebraic group $A(G)$ with $\tau(A)$ normalizing $\tau(B)$, and hence their product $\tau(A)\tau(B) = \tau(AB)$ is an algebraic subgroup of $A(G)$. Consequently, AB is an algebraic subgroup of G by Proposition 5.1. ∎

5.2 Extension Lemma

The main purpose of this section is to prove a key lemma on the extension of analytic representations and representative functions of certain normal subgroups for later use. We begin with the following technical lemma on Lie algebras.

Lemma 5.3 *Let* $\mathfrak{g}$ *be a (finite-dimensional) Lie algebra over a field of characteristic* 0*, and assume that* $\mathfrak{g}$ *is a semidirect sum* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{k}$*, where* $\mathfrak{k}$ *is a solvable ideal of* $\mathfrak{g}$*, and* $\mathfrak{a}$ *is a complementary subalgebra that is reductive in* $\mathfrak{g}$*. If* $\mathfrak{m} = [\mathfrak{g}\ ,\mathfrak{k}]$*, then* $\mathfrak{k}$ *contains a nilpotent subalgebra* $\mathfrak{h}$ *such that* $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ *(not necessarily semidirect sum) and that* $\mathfrak{h}$ *centralizes* $\mathfrak{a}$*.*

Proof. Let $\mathfrak{z}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$; that is,

$$\mathfrak{z} = \{x \in \mathfrak{k} : [x, \mathfrak{a}] = (0)\}.$$

Then $\mathfrak{k} = \mathfrak{z} + \mathfrak{m}$. In fact, since $ad(a)(\mathfrak{k}) \subset \mathfrak{m}$ for all $a \in \mathfrak{a}$, $\mathfrak{m}$ is $ad(\mathfrak{a})$-stable. Since $\mathfrak{k}$ is semisimple as an $\mathfrak{a}$-module under the adjoint representation, there exists an $ad(\mathfrak{a})$-stable subspace B of $\mathfrak{k}$ such that $\mathfrak{k} = B \oplus \mathfrak{m}$ (as $\mathfrak{a}$-modules). Now $[\mathfrak{a}, B] \subset B \cap [\mathfrak{g}, \mathfrak{k}] = (0)$ implies $B \subset \mathfrak{z}$, and $\mathfrak{k} = \mathfrak{z} + \mathfrak{m}$ follows. If $x \in \mathfrak{z}$ is a regular element of the Lie algebra $\mathfrak{z}$, then

$$\mathfrak{h} = \{y \in \mathfrak{z} : ad(x)^i(y) = 0 \text{ for some } i\}$$

is a Cartan subalgebra of $\mathfrak{z}$, and we have the decomposition $\mathfrak{z} = \mathfrak{h} + \mathfrak{c}$, where $\mathfrak{c}$ is a subspace of $\mathfrak{z}$ with $ad(x)(\mathfrak{c}) = \mathfrak{c}$ (see §A.4). Hence $\mathfrak{c} \subset \mathfrak{m}$, and $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ follows. ∎

Lemma 5.4 *Let F be a closed normal complex analytic subgroup of a faithfully representable complex analytic group G, and assume that G/F is isomorphic with a complex vector group. Let $N(G)$ denote the representation radical of G, and H a maximal reductive subgroup of G. Then*

(i) *$HN(G)$ is a closed normal subgroup of G;*

(ii) *$HN(G) \subset F$;*

(iii) *Every $N(G)$-unipotent complex analytic representation of F extends to an $N(G)$-unipotent complex analytic representation of G.*

In particular, the restriction map $R(G, N(G))(= R(G)) \to R(F, N(G))$ is surjective.

Proof. Let $N = N(G)$.

(i) Let $\pi : G \to G/N$ be the canonical morphism. By Theorem 4.38 and Proposition 4.42, we have $G/N = U \times D$, where U is a complex vector subgroup, and D is a reductive subgroup, which is necessarily the (unique) maximal reductive subgroup of G/N. Thus we have $HN = \pi^{-1}(D)$ by Proposition 4.41, and since D is closed and normal in G/N, (i) follows.

(ii) Since G/F is abelian, the commutator subgroup $G' \subset F$, and N is contained in F as N is the radical of G' (Corollary 4.39). Consider the isomorphism

$$HF/F \cong H/(H \cap F).$$

On the one hand, HF/F is a complex vector group as a complex analytic subgroup of the complex vector group G/F. On the other hand, the abelian group $H/(H \cap F)$ is a complex torus as a homomorphic image of the reductive group H. Thus HF/F is trivial, and $H \subset F$ follows. This shows that $HN \subset F$, proving (ii).

(iii) Write G as a semidirect product $G = H \cdot K$, where K is a nucleus of G. By Lemma 5.3, K contains a simply connected nilpotent complex analytic subgroup P such that $K = PN$ and that P centralizes H. Then $G = PNH = PF$, and clearly we have $[P, G] \subset N$. Since G/F is a vector group and $G = PF$, we can find complex one-parameter subgroups $P_1, \cdots, P_r \subset P$ such that G is expressed in successive semidirect products $G = P_r \cdots P_1 \cdot F$.

Let $D_0 = F$, and define $D_{i+1} = P_{i+1} \cdot D_i$ (semidirect product), where $0 \le i \le r-1$. The assertion (iii) follows as soon as we have shown that every N-unipotent complex analytic representation of D_i, $0 \le i \le r-1$, is extendable to an N-unipotent complex analytic representation of D_{i+1}. Let ρ be an N-unipotent of D_i. Since $[P, G] \subset N$, the action of the group P_{i+1} on D_i/N is trivial. Applying Theorem 3.7 to the semidirect product $D_{i+1} = P_{i+1} \cdot D_i$, we see that ρ has an N-unipotent extension σ to D_{i+1}. This completes the proof of (iii).

The surjectivity of $R(G, N) \to R(F, N)$ follows from (iii) and Lemma 3.5. ■

Corollary 5.5 *Under the hypothesis of Lemma 5.4, every N-unipotent complex analytic representation of HN extends to an N-unipotent complex analytic representation of G, and the restriction map*

$$R(G) = R(G, N) \to R(HN, N)$$

is surjective.

Proof. Since G/HN is a complex vector group, the assertion follows from Lemma 5.4. ■

Corollary 5.6 *Under the hypothesis of Lemma 5.4, there exist complex one-parameter subgroups $P_1, \cdots, P_s$ of F such that*

$$R(F, N) \cong R(P_s) \otimes \cdots \otimes R(P_1) \otimes R(HN, N).$$

Proof. By Lemma 5.4, $HN \subset F$, and using the notation in the proof of Lemma 5.4, we have $F = (P \cap F)HN$. Using the same argument as in the proof of Lemma 5.4 (iii), we may find complex one-parameter subgroups $P_1, \cdots, P_s$ of $F \cap P$ such that F is expressed in successive semidirect products $F = P_s \cdots P_1 \cdot (HN)$ and that, for all $1 \le i \le s$, the restriction map

$$R(P_i \cdots P_1(HN), N) \to R(P_{i-1} \cdots P_1(HN), N)$$

is surjective. We then have

$$R(P_i \cdots P_1 \cdot (HN), N) \cong R(P_i) \otimes R(P_{i-1} \cdots P_1 \cdot (HN), N)$$

for all i, and, putting these isomorphisms together, we obtain

$$R(F, N) \cong R(P_s) \otimes \cdots \otimes R(P_1) \otimes R(HN, N).$$

■

Corollary 5.7 *Under the hypothesis of Lemma 5.4, the* $\mathbb{C}$*-algebra* $R(F, N)$ *is not finitely generated unless* $F = HN$.

Proof. Since the $\mathbb{C}$-algebra $R(\mathbb{C})$ is *not* finitely generated (Example 2.19), the assertion follows from Corollary 5.6. ■

5.3 Affine Algebraic Structure on Reductive Groups

Tannaka Duality We first review certain operations on matrices. For any two square matrices A and B of degree m and n, respectively, we define

(i) $A \dot{+} B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, the direct sum of A and B;

(ii) $A \otimes B = \begin{pmatrix} a_{11}B & \cdot & \cdot & a_{1n}B \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1}B & \cdot & \cdot & annB \end{pmatrix}$, the tensor product of A and B.

Let G be a real analytic group, and let $\mathcal{R}$ denote the set Rep(G) of all analytic (matrix) representations of G over $\mathbb{C}$, i.e., analytic homomorphisms ρ from G into $GL(n, \mathbb{C})$ for some positive integer n, the *degree* of ρ, which is denoted by $d(\rho)$. Recall (§2.1) that if $\rho_i : G \to GL(V_i, \mathbb{C})$ is a representation of a group G on a $\mathbb{C}$-linear space V_i, $i = 1, 2$, then $\rho_1 \oplus \rho_2$ and $\rho_1 \otimes \rho_2$ are representations on the spaces $V_1 \oplus V_2$ and $V_1 \otimes V_2$, respectively. If the representation

$$\rho_i : G \to GL(n_i, \mathbb{C})$$

is a matrix representation ($i = 1, 2$), then $\rho_1 \oplus \rho_2$ and $\rho_1 \otimes \rho_2$ are the matrix representations, defined by

$$(\rho_1 \oplus \rho_2)(x) = \rho_1(x) \dot{+} \rho_2(x),$$

and

$$(\rho_1 \otimes \rho_2)(x) = \rho_1(x) \otimes \rho_2(x)$$

for each $x \in G$.

By a *representation* of $\mathcal{R}$, we shall mean a map

$$\zeta : \mathcal{R} \to \bigcup_n GL(n, \mathbb{C})$$

which satisfies

(i) $\zeta(\rho) \in GL(d(\rho), \mathbb{C})$;

(ii) $\zeta(\rho \oplus \sigma) = \zeta(\rho) \dot{+} \zeta(\sigma)$;

(iii) $\zeta(\rho \otimes \sigma) = \zeta(\rho) \otimes \zeta(\sigma)$;

(iv) $\zeta(\gamma\rho\gamma^{-1}) = \gamma\zeta(\rho)\gamma^{-1}$

for $\rho, \sigma \in \mathcal{R}$, and any nonsingular matrix γ of degree $d(\rho)$.

The set $G^{\mathbb{C}}(\mathcal{R})$ of all representations of $\mathcal{R}$ becomes a group under the operation

$$(\zeta_1, \zeta_2) \mapsto \zeta_1 \cdot \zeta_2 : G^{\mathbb{C}}(\mathcal{R}) \times G^{\mathbb{C}}(\mathcal{R}) \to G^{\mathbb{C}}(\mathcal{R})$$

where $(\zeta_1 \cdot \zeta_2)(\rho) = \zeta_1(\rho)\zeta_2(\rho)$ for all $\rho \in \mathcal{R}$. We topologize $G^{\mathbb{C}}(\mathcal{R})$ with the weakest topology, which makes the map

$$\zeta \mapsto \zeta(\rho) : G^{\mathbb{C}}(\mathcal{R}) \to GL(d(\rho), \mathbb{C})$$

continuous for each $\rho \in \mathcal{R}$. The group $G^{\mathbb{C}}(\mathcal{R})$ with this topology becomes a topological group. The set

$$G(\mathcal{R}) = \{\zeta \in G^{\mathbb{C}}(\mathcal{R}) : \zeta(\overline{\rho}) = \overline{\zeta(\rho)}\}$$

forms a subgroup of $G^{\mathbb{C}}(\mathcal{R})$. For $x \in G$, define the representation ζ_x of $\mathcal{R}$ by $\zeta_x(\rho) = \rho(x)$ for all $\rho \in \mathcal{R}$. Then $x \mapsto \zeta_x$ defines a canonical (group) homomorphism $\theta : G \to G^{\mathbb{C}}(\mathcal{R})$, and clearly we have $\theta(G) \subset G(\mathcal{R})$.

In the following discussion, we use the notation ρ_{ij} for any matrix representation ρ of G to denote the (i,j)-coefficient function of ρ, i.e., the function which maps $x \in G$ to the (i,j)-entry $\rho_{ij}(x)$ of $\rho(x)$. The $\mathbb{C}$-algebra $R(G)$ of all analytic representative functions of G is spanned by the coefficient functions ρ_{ij}, $(1 \le i, j \le d(\rho))$, $\rho \in \mathcal{R}$. Let $\Omega = Hom_{\mathbb{C}-alg}(R(G), \mathbb{C})$. Then Ω is a group under the convolution and is given with the structure of a pro-affine algebraic group for which $R(G)$ is the polynomial algebra. For $\phi \in \Omega$, define ζ_ϕ to be the representation of $\mathcal{R}$, which maps $\rho \in \mathcal{R}$ to the matrix whose (i,j)

entry is $\phi(\rho_{ij})$. The map $\phi \mapsto \zeta_\phi$ defines a canonical homomorphism $\chi : \Omega \to G^{\mathbb{C}}(\mathcal{R})$. Since the coefficient functions $\rho_{i,j}$ span $R(G)$, it is clear that this map is an injection. We have a commutative diagram

$$\begin{array}{ccccc} & & G & & \\ & \swarrow^{\tau} & \downarrow & \searrow^{\theta} & \\ A(G) & \xrightarrow[\omega]{} & \Omega & \xrightarrow[\chi]{} & G^{\mathbb{C}}(\mathcal{R}) \end{array} \tag{5.3.1}$$

where ω is the canonical isomorphism (2.5.1) of §2.5, and the vertical map $G \to \Omega$ is the canonical injection, which maps each $x \in G$ to the evaluation homomorphism $f \mapsto f(x)$, $f \in R(G)$.

Suppose that G is a compact real analytic group. G admits a (finite-dimensional) faithful analytic representation φ (see, e.g., [4], Theorem 4, p. 211). We may assume that φ is a representation over $\mathbb{C}$, i.e., the representation space of φ is a $\mathbb{C}$-linear space. φ and its dual $\varphi^\circ (= \overline{\varphi})$ generate all real analytic representations of G (see, e.g., [4], Proposition 3, p. 190), and hence the bistable finite-dimensional $\mathbb{C}$-linear subspace $S = [\varphi] + \overline{[\varphi]}$ generates the $\mathbb{C}$-algebra $R(G)$. In particular, $R(G)$ is finitely generated, and hence the pro-affine algebraic group Ω is affine. The injection $\chi : \Omega \to G^{\mathbb{C}}(\mathcal{R})$ becomes an isomorphism ([4], Proposition 2, p. 196), and thus the composite map $\chi \circ \omega : A(G) \to G^{\mathbb{C}}(\mathcal{R})$ in the diagram (5.3.1) is an isomorphism. Chevalley's formulation of Tannaka's theorem on duality ([4], Th. 5, p. 211) states:

Theorem 5.8 *For a compact real analytic group G, the canonical map $\theta : G \to G^{\mathbb{C}}(\mathcal{R})$ defines an isomorphism $\theta : G \cong G(\mathcal{R})$ of Lie groups.* ∎

On the other hand, the universal algebraic hull $A(G)$ is a linear complex analytic group by Theorem 2.23, and if we define

$$B(G) = \{\alpha \in A(G) : \overline{\alpha(f)} = \alpha(\overline{f}) \text{ for all } f \in R(G)\},$$

$B(G)$ is a closed subgroup of $A(G)$. Since the left translation of $R(G)$ by an element of G commutes with the conjugation, clearly the canonical injection $\tau : G \to A(G)$ maps G into $B(G)$. The Hochschild-Mostow's formulation of Tannaka's theorem on duality ([12], Th. 5.1, p. 516) states:

Theorem 5.9 *If G is a compact real analytic group, then the canonical map $\tau : G \to A(G)$ defines an isomorphism $\tau : G \cong B(G)$ of Lie groups.* ∎

We note that the two versions of Tannaka's Theorem stated above are equivalent by virtue of the isomorphism $\chi \circ \omega : A(G) \to G^{\mathbb{C}}(\mathcal{R})$ in the diagram (5.3.1).

Affine Structure on Reductive Groups The following result enables us to equip a reductive complex analytic group with the structure of an affine algebraic group.

Theorem 5.10 *Let G be a compact real analytic group. Then the complex analytic group $A(G)$ together with the canonical injection $\tau : G \to A(G)$ is the universal complexification of G.*

Proof. Let $\gamma : G \to G^+$ and $\tau : G \to A(G)$ be the canonical injections of the compact group G into the universal complexification and the universal algebraic hull of G, respectively. G^+ is a reductive complex analytic group by Theorem 4.29. We view the reductive complex Lie group G^+ as a linear group, say $G^+ \subset GL(W, \mathbb{C})$ for some finite-dimensional linear space W, and the map γ as an analytic representation $\gamma : G \to GL(W, \mathbb{C})$. By the universal property of $\tau : G \to A(G)$ (Proposition 2.21), there is a rational representation $\widetilde{\gamma} : A(G) \to GL(W, \mathbb{C})$ such that $\widetilde{\gamma} \circ \tau = \gamma$. Also by the universal property of the canonical map $\gamma : G \to G^+$ (see §1.5), there exists a complex analytic homomorphism

$$\tau^+ : G^+ \to A(G)$$

such that $\tau^+ \circ \gamma = \tau$. We claim that τ^+ is an isomorphism with inverse $\widetilde{\gamma}$. We have

$$(\widetilde{\gamma} \circ \tau^+) \circ \gamma = \widetilde{\gamma} \circ (\tau^+ \circ \gamma) = \widetilde{\gamma} \circ \tau = \gamma; \qquad (5.3.2)$$

$$(\tau^+ \circ \widetilde{\gamma}) \circ \tau = \tau^+(\widetilde{\gamma} \circ \tau) = \tau^+ \circ \gamma = \tau. \qquad (5.3.3)$$

(5.3.2) shows that $\widetilde{\gamma} \circ \tau^+ = 1$ on $\gamma(G)$, and since $\mathcal{L}(\gamma(G))$ is a real form of $\mathcal{L}(G^+)$, it follows that $\widetilde{\gamma} \circ \tau^+ = 1_{G^+}$.

Next we show that $\tau^+ \circ \widetilde{\gamma} = 1_{A(G)}$. We have $\tau^+ \circ \widetilde{\gamma} = 1$ on $\tau(G) = B(G)$ by (5.3.3). Hence it is enough to show that the real Lie algebra $\mathcal{L}(B(G))$ spans $\mathcal{L}(A(G))$ over $\mathbb{C}$. Let S be a bistable finite-dimensional subspace of $R(G)$ such that S generates the algebra $R(G)$, and let

$$\rho : G \to GL(S, \mathbb{C})$$

be the representation by left translations on S. Let

$$T = \{f \in S : f = \overline{f}\}.$$

T is a real form of the complex linear space S, i.e., $S = T \oplus \sqrt{-1}T$. The isomorphism $\alpha \mapsto \alpha_S : A(G) \to A(G)_S$ (Theorem 2.23) maps $B(G)$ onto $G_S = \{\alpha_S : \alpha \in A(G), \alpha(T) = T\}$, and thus we have the commutative diagram

$$\begin{array}{ccc} B(G) & \xrightarrow{\subseteq} & A(G) \\ \cong \downarrow & & \downarrow \cong \\ G_S & \xrightarrow[\subseteq]{} & A(G)_S \end{array}$$

The map ρ induces an isomorphism $G \cong G_S = \mathrm{Im}(\rho)$ by Tannaka's theorem (Theorem 5.9). G_S is a real algebraic subgroup of $GL(S, \mathbb{C})$ and is the image of G_T under the natural embedding

$$\mu : GL(T, \mathbb{R}) \to GL(S, \mathbb{C}).$$

We identify the real algebraic subgroups G_S and G_T under μ. G_S is Zariski dense in $A(G)_S$, and the algebraic group $A(G)_S$ is the extension (over $\mathbb{C}$) of the real algebraic group $G_S (= \mu(G_T))$. Hence $\mathcal{L}(A(G)_S)$ is spanned by the real Lie algebra $\mathcal{L}(G_S)$ over $\mathbb{C}$ (see, e.g., [5], Prop. 2, p. 129). It follows from the commutative diagram above that $\mathcal{L}(A(G))$ is spanned by the real Lie subalgebra $\mathcal{L}(B(G))$ over $\mathbb{C}$. This establishes $\tau^+ \circ \widetilde{\gamma} = 1_{A(G)}$, and hence completes our proof. ∎

Theorem 5.11 *Every reductive complex analytic group H has a unique affine algebraic group structure in such a way that the complex analytic representations of H are exactly the rational representations of H.*

Proof. Let Q be a maximal compact subgroup of H. The inclusion $\iota : Q \to H$ is the universal complexification of the compact analytic group Q by Theorem 4.31, and hence we have

$$A(Q) \cong Q^+ \cong H$$

by Theorem 5.10. We may therefore identify the complex analytic group H with $A(Q)$, and equip H with the affine algebraic group structure of $A(Q)$. Suppose $\rho : H \to GL(V, \mathbb{C})$ is a complex analytic representation of H, and let $\sigma = \rho|_Q$. The analytic representation σ of Q can be then extended to a rational representation

$$\widetilde{\sigma} : A(Q) = H \to GL(V, \mathbb{C}),$$

i.e., $\sigma = \widetilde{\sigma} \circ \iota$. On the other hand, since $\iota : Q \to A(Q) = H$ is the universal complexification, the uniqueness of the induced complex analytic representation

$$\sigma^+ : H \to GL(V, \mathbb{C})$$

with $\sigma^+ \circ \iota = \sigma$ implies $\rho = \widetilde{\sigma}$, proving that ρ is rational.

Since every complex analytic representation of H is rational, the polynomial functions on H are exactly the representative functions on the analytic group $Q^+ = H$, and therefore the algebraic group structure on H is necessarily unique. ■

Lemma 5.12 *If K is a closed reductive complex Lie subgroup of a faithfully representable complex analytic group G such that K/K_0 is finite, then K is an algebraic subgroup of G.*

Proof. Let H be a maximal reductive complex Lie subgroup of G that contains K. Then H is connected, and G is a semidirect product $G = H \cdot S$, where S is a nucleus of G (Theorem 4.43). We first show that H is an algebraic subgroup of G. Every complex analytic representation of H extends in an obvious way to a complex analytic representation of G, and hence the restriction map $R(G) \to R(H)$ is surjective, i.e., $R(H) = R(G)_H$. $R(H)$ is finitely generated by Corollary 4.32, and the canonical monomorphism

$$H \to A(H) = Aut_H(R(H))$$

is an isomorphism (Theorem 5.11). This shows that H is an algebraic subgroup of G. Next we show K is an algebraic subgroup of G. Since K/K_0 is finite, it is enough to show that K_0 is an algebraic subgroup of H. Thus we may assume that K itself is connected. By Theorem 5.11, the reductive complex analytic group K has a unique algebraic group structure in such a way that the complex analytic representations of K are exactly the rational representations. Hence if we view the affine algebraic group H as a linear algebraic group and the inclusion map $j : K \to H$ as a linear analytic representation, then j is rational, and $K = j(K)$ is Zariski closed in the algebraic group H. This shows that the restriction map $R(H) \to R(K)$ is surjective. Since we already have $R(H) = R(G)_H$, $R(K) = R(G)_K$, and this proves that K is an algebraic subgroup of G. ■

5.4 The Maximum Algebraic Subgroup

We first prove that the representation radical $N(G)$ of a faithfully representable complex analytic group G is an algebraic subgroup. We begin with

Lemma 5.13 *Let N be a unipotent complex analytic subgroup of a full complex linear group $GL(V,\mathbb{C})$, and let H be a complex analytic subgroup of $GL(V,\mathbb{C})$ such that H normaalizes N. For $h \in H$, let $\kappa(h)$ denote the automorphism of N given by $n \mapsto hnh^{-1}$. Then for each $f \in R(N,N)$, the set $\{f \circ \kappa(h) : h \in H\}$ spans a finite-dimensional subspace of $R(N,N)$.*

Proof. We note that by Lemma 3.1, $f \circ \kappa(H) \subset R(N,N)$. Let $h \in H$. By Corollary 3.11, $\kappa(h) : N \to N$ is a polynomial map of degree $\leq m^2$, where $m = \dim(V)$. Thus if f is a polynomial function of N of degree $\leq n$, then $f \circ \kappa(h)$ is a polynomial function of degree $\leq nm^2$. This shows that the set $f \circ \kappa(H)$ spans a finite-dimensional space. ■

Proposition 5.14 *If G is a faithfully representable complex analytic group, then its representation radical $N(G)$ is a unipotent algebraic subgroup of G with polynomial algebra $R(N(G), N(G))$.*

Proof. Put $N = N(G)$, and choose a faithful complex analytic representation $\tau : G \to GL(W,\mathbb{C})$. Then $N = rad(G')$ by Corollary 4.39, and N is a simply connected nilpotent analytic group (Theorem 4.38). The (analytic group) isomorphism $N \cong \tau(N)$ induces a $\mathbb{C}$-algebra isomorphism

$$R(N,N) \cong R(\tau(N), \tau(N)).$$

On the other hand, $\tau(N)$ is a unipotent subgroup of $GL(W, \mathbb{C})$ by Theorem 4.38, and as such it is an algebraic (i.e., Zariski closed) subgroup of $GL(W, \mathbb{C})$ by Proposition 3.12. Now $R(\tau(N), \tau(N))$ is the polynomial algebra of the algebraic group $\tau(N)$ by Theorem 3.14, and this, in particular, implies that $R(N, N) \cong R(\tau(N), \tau(N))$ is finitely generated. From the isomorphisms

$$N \cong \tau(N) \cong Aut_{\tau(N)}(R(\tau(N), \tau(N))) \cong Aut_N(R(N, N)),$$

we deduce that the canonical map

$$N \to Aut_N(R(N, N)).$$

is an isomorphism, i.e., N is an affine algebraic group with the polynomial algebra $R(N, N)$. Thus to prove that N is an algebraic subgroup of G, it remains to show

$$R(N, N) = R(G)_N, \tag{5.4.1}$$

i.e., the restriction morphism $R(G) \to R(N, N)$ is surjective. Let H be a maximal reductive complex analytic subgroup of G. Then the restriction map

$$R(\tau(H)\tau(N), \tau(N)) \to R(\tau(N), \tau(N))$$

is surjective by Lemma 3.2 and Lemma 5.13, and hence the map $R(HN, N) \to R(N, N)$ is also surjective. Since the restriction map $R(G) \to R(HN, N)$ is surjective by Corollary 5.5, (5.4.1) follows. ■

Corollary 5.15 *If G is a faithfully representable complex analytic group, its commutator subgroup G' is an algebraic subgroup of G.*

Proof. By Corollary 4.39, $N(G)$ is the radical of G', and hence $G' = N(G)S$, where S is a maximal semisimple complex analytic subgroup of G'. $N(G)$ is algebraic in G by Proposition 5.14, and S is also algebraic in G as a reductive complex analytic subgroup of G (Lemma 5.12). Thus we see that $G' = N(G)S$ is an algebraic subgroup of G by Remark 5.2. ■

We are ready to present the main result in this chapter.

Theorem 5.16 *Let G be a faithfully representable complex analytic group and let H be a maximal reductive complex analytic subgroup of G. Then*

(i) *The complex analytic subgroup $HN(G)$, which is independent of maximal reductive subgroups H, is an algebraic subgroup of G, and is in fact the maximal such in the sense that $HN(G)$ contains all algebraic subgroups of G.*

(ii) *Every complex analytic representation of G induces a rational representation of $HN(G)$, and, conversely, every rational representation of $HN(G)$ is obtained in this way.*

Proof. Let $\tau : G \to A(G)$ denote the canonical injection, and let $N = N(G)$. H and N are algebraic subgroups of G by Lemma 5.12 and Proposition 5.14, respectively. Hence $\tau(N)$ and $\tau(H)$ are algebraic (i.e., Zariski closed) subgroups of the pro-affine algebraic group $A(G)$ by Proposition 5.1. It follows that $\tau(HN) = \tau(N)\tau(H)$ is Zariski closed in the pro-affine algebraic group $A(G)$, proving that HN is an algebraic subgroup of G again by Proposition 5.1. That the subgroup HN is independent of maximal reductive subgroups H follows from Theorem 4.48. We now show that HN is the maximum algebraic subgroup of G. Let D be any algebraic subgroup of G. We claim $D \subset HN$. Suppose D is *not* contained in HN. We first show that there is a complex analytic representation φ of G such that $\varphi(D)$ is a nontrivial *unipotent* subgroup. Let $\pi : G \to G/HN$ be the canonical morphism, and let

$$\rho : G \to GL(V, \mathbb{C})$$

be a faithful complex analytic representation of G. $\rho(HN)$ is an algebraic subgroup of $GL(V, \mathbb{C})$ by Proposition 5.1. Since HN is normal in G, the algebraic subgroup $\rho(HN)$ of $GL(V, \mathbb{C})$ is normal in $\rho(G)$ and hence also in the Zariski closure $\rho(G)^*$ of $\rho(G)$ in $GL(V, \mathbb{C})$. Thus $\rho(G)^*/\rho(HN)$ is a linear algebraic group, and the composite map

$$G/HN \to \rho(G)/\rho(HN) \xrightarrow{\subseteq} \rho(G)^*/\rho(HN)$$

defines a faithful complex analytic representation $\widetilde{\rho}$ of G/HN. Let $\varphi = \widetilde{\rho} \circ \pi$. Since D is algebraic in G, $\varphi(D)$ is Zariski closed in the algebraic group $\rho(G)^*/\rho(HN)$ by Proposition 5.1. Since $\pi(D)$ is a nontrivial subgroup of the vector group G/HN with finitely many connected components, $\pi(D)$ is a nontrivial vector subgroup of G/HN. Thus $\varphi(D) = \widetilde{\rho}(\pi(D))$ is a complex vector subgroup of the abelian algebraic group $\widetilde{\rho}(\pi(G))^*$, and hence the algebraic subgroup

$\varphi(D)$ is unipotent. Next we choose a nontrivial *semisimple* complex analytic representation ψ of G. Such a representation always exists, namely, the composite of the natural morphism $G \to G/HN$ and a faithful, semisimple complex analytic representation of the complex vector group G/HN (Example 2.6). If γ denotes the direct sum of the representations φ and ψ, then γ is a complex analytic representation of G such that $\gamma(D)$ is *not* Zariski closed, and this contradicts the assumption that D is an algebraic subgroup of G. This shows that $D \subset HN$, and (i) follows.

We now prove (ii). Let ρ be a rational representation of the algebraic group HN. Since N is a unipotent algebraic subgroup of HN, ρ is an N-unipotent analytic representation of the analytic group HN, and Corollary 5.5 enables us to extend ρ to a complex analytic representation of G. ■

The algebraic subgroup $HN(G)$ of G in Theorem 5.16 is called the *maximum algebraic subgroup* of G, and we denote it by $\mathcal{M}(G)$.

Remark 5.17 $P(HN) = R(HN, N)$, i.e., the polynomial algebra of the maximum algebraic subgroup HN of G in Theorem 5.16 is the algebra of the representative functions associated with all N-unipotent complex analytic representations of G. In fact, from $P(H) = R(G)_H = R(H)$ and $P(N) = R(G)_N = R(N, N)$, we obtain

$$P(HN) \cong P(H) \otimes P(N) = R(H) \otimes R(N, N) \cong R(HN, N),$$

where the first isomorphism above follows from standard properties of algebraic groups (see §B.2), and the last one follows from Theorem 3.3. ■

Remark 5.18 The representation radical N is easily seen to be the unipotent radical of the algebraic group HN. ■

Proposition 5.19 *If L is a complex analytic subgroup of a faithfully representable complex analytic group G, then any algebraic subgroup of L is algebraic in G. In particular, every algebraic subgroup of L is an algebraic (i.e., Zariski closed) subgroup of the algebraic group $\mathcal{M}(G)$.*

Proof. Let K be an algebraic subgroup of L, and we prove that K is an algebraic subgroup of G. For that it is enough to show $P(K) = R(G)_K$. Let H be a maximal reductive subgroup of G and let $N = N(G)$ so that $\mathcal{M}(G) = HN$ (Theorem 5.16), and choose a faithful complex analytic representation $\rho : G \to GL(V, \mathbb{C})$. By Proposition 5.1 applied to ρ and $\rho_L : L \to GL(V, \mathbb{C})$, we see that both $\rho(HN)$ and $\rho(K)$ are algebraic subgroups of $GL(V, \mathbb{C})$ in the usual sense. Thus $\rho(K)$ is an algebraic subgroup of the algebraic group $\rho(HN)$, and hence the restriction map

$$P(\rho(HN)) \longrightarrow P(\rho(K))$$

is surjective. Since ρ induces isomorphisms $HN \cong \rho(HN)$ and $K \cong \rho(K)$ of algebraic groups, the restriction map $P(HN) \to P(K)$ is surjective. On the other hand, we have $P(HN) = R(HN, N)$ (Remark 5.18), and since K is an algebraic subgroup of L, we have $P(K) = R(L)_K$. Consequently

$$R(HN, N) \longrightarrow R(L)_K$$

is surjective. Since the map $R(G) \to R(HN, N)$ is surjective by Corollary 5.5, the map $R(G) \to R(L)_K$ is surjective.

The second assertion of the proposition follows from the first one and Theorem 5.16. ∎

In the next theorem we determine when a faithfully representable complex analytic group G itself is its own algebraic subgroup, that is, $\mathcal{M}(G) = G$.

Theorem 5.20 *Let G be a faithfully representable complex analytic group. Then the following are equivalent.*

(i) *G admits the structure of an affine algebraic group which is compatible with the structure of the analytic group of G in the sense that the complex analytic representations are exactly the rational representations of G.*

(ii) *$G = H \cdot N(G)$ (semidirect product), where H is a maximal reductive subgroup of G.*

(iii) *$R(G)$ is finitely generated.*

(iv) *The canonical injection $\tau : G \to A(G)$ is an isomorphism.*

Proof. (i) and (ii) are equivalent by Theorem 5.16. If (i) holds, $P(G) = R(G)$ and hence (iii) follows, and (iii) $\Rightarrow$(ii) by Corollary 5.7. Thus (i), (ii), and (iii) are equivalent. (iv) $\Rightarrow$ (i) by Proposition 5.1. To show (i)$\Rightarrow$(iv), we first note that (i) is equivalent to the statement that G itself is the algebraic subgroup of G. Thus (i) implies that $\tau(G)$ is an algebraic subgroup of $A(G)$ by Proposition 5.1. Since $\tau(G)$ is always Zariski dense in $A(G)$, we get $A(G) = \tau(G)$, proving (iv). ∎

Remark 5.21 Suppose that G is a complex analytic subgroup of a full linear group $GL(V, \mathbb{C})$, and assume $R(G)$ is finitely generated. Then G is an algebraic subgroup of the algebraic group $GL(V, \mathbb{C})$ (i.e., G is Zariski closed in $GL(V, \mathbb{C})$), and its polynomial algebra is $R(G)$. In fact, since $R(G)$ is finitely generated, G admits the structure of an affine algebraic group with $R(G)$ as its polynomial algebra by Theorem 5.20. On the other hand, the canonical map $\tau : G \to A(G)$ is an isomorphism by Theorem 5.20, and the inclusion $\iota : G \to GL(V, \mathbb{C})$ induces a rational representation of $A(G)$

$$\widetilde{\iota} : A(G) \to GL(V, \mathbb{C})$$

such that $\widetilde{\iota} \circ \tau = \iota$. Then $G = \iota(G) = \widetilde{\iota}(A(G))$ is Zariski closed in the algebraic group $GL(V, \mathbb{C})$. The polynomial algebra $P(G)$ of this linear algebraic subgroup G is clearly contained in $R(G)$. This means that the identity map

$$i_G : (G, P(G)) \to (G, R(G))$$

is a morphism of affine algebraic groups. Since a bijective morphism is an isomorphism in affine algebraic groups, i_G is an isomorphism of algebraic groups, proving $P(G) = R(G)$. ∎

Remark 5.21, together with Theorem 5.20, yields the following.

Corollary 5.22 *Let G be a complex analytic subgroup of a full linear group $GL(V, \mathbb{C})$, and assume that $R(G)$ is finitely generated. Then G is an algebraic subgroup of $GL(V, \mathbb{C})$, and any complex analytic representation of G is rational.*

Remark 5.23 Any linear semisimple complex analytic group G, for example, is an algebraic group by Corollary 5.22, and every complex analytic representation of G is rational. However, this is not the case, in general, for linear semisimple *real* analytic groups. To construct such an example, let Ad denote the adjoint representation of the real semisimple group $SL(3, \mathbb{R})$:

$$Ad : SL(3, \mathbb{R}) \longrightarrow GL(\mathrm{sl}(3, \mathbb{R}), \mathbb{R}) = GL(8, \mathbb{R})$$

and let G denote the semisimple analytic subgroup $\mathrm{Im}(Ad)$ of $GL(8, \mathbb{C})$. We claim that G is a desired example. In fact, assume the contrary. Thus G is an algebraic subgroup of $SL(3, \mathbb{R})$, and every continuous representation of G is rational. Since Ad maps $SL(3, \mathbb{R})$ onto G homeomorphically, we may consider the continuous (and hence real analytic) representation ρ of G which is the composition

$$G \stackrel{Ad^{-1}}{\longrightarrow} SL(3, \mathbb{R}) \subset GL(3, \mathbb{R}).$$

Then ρ is a rational representation of G by assumption, and hence

$$Ad : SL(3, \mathbb{R}) \to G$$

becomes an isomorphism of real algebraic groups. This implies that the complexification $SL(3, \mathbb{C})$ of $SL(3, \mathbb{R})$ is isomorphic with the complexification $Ad_{\mathbb{C}}(SL(3, \mathbb{C}))$ of G, where $Ad_{\mathbb{C}}$ denotes the adjoint representation of $SL(3, \mathbb{C})$.

On the other hand, the two groups $SL(3, \mathbb{R})$ and $Ad_{\mathbb{C}}(SL(3, \mathbb{C}))$ cannot be isomorphic: the center of $SL(3, \mathbb{R})$ is of order 3 while $Ad_{\mathbb{C}}(SL(3, \mathbb{C}))$ has the trivial center. ■

5.5 Further Properties of Reductive Groups

We establish further results on complex reductive groups making use of their affine algebraic structure.

Theorem 5.24 *Let G be a complex analytic group, and K be a closed normal complex analytic subgroup of G. If any two of the groups G, K, and G/K are reductive, then so is the remaining group.*

Proof. (i) We have already established in Theorem 4.35 that if K and G/K are reductive, then so is G.

(ii) Now we assume that G and G/K are reductive, and we prove that K is reductive. By Theorem 5.11, G may be identified with a reductive algebraic linear group such that every complex analytic representation of G is a rational representation. Since G/K has a faithful complex analytic representation as a reductive group, ρ say, then the composition of ρ with the canonical map $G \to G/K$ is a complex analytic (and hence also rational) representation of G, and its kernel K is therefore a normal algebraic subgroup of the reductive algebraic group G. Thus K is also a reductive algebraic linear group, and therefore is a reductive complex analytic group.

(iii) Finally we assume that G and K are reductive, and prove that G/K is reductive. Since the group G is reductive, each complex analytic representation of G is semisimple, and hence the same holds for G/K. Thus to show that G/K is reductive, it remains to show that G/K is faithfully representable. Let ρ be any faithful complex analytic representation of G. Then ρ is a rational representation of the algebraic group G by Theorem 5.20, and $\rho(G)$ is, therefore, a linear algebraic group. Since K is an algebraic subgroup of G (Lemma 5.12), $\rho(K)$ is a normal algebraic subgroup of $\rho(G)$ (cf., Proposition 5.1), and hence by ([6], Prop. 11, p. 119), there is a rational representation σ of $\rho(G)$ whose kernel is exactly $\rho(K)$. The representation $\sigma \circ \rho$ of G is a complex analytic representation whose kernel is K, and it induces a faithful complex analytic representation of G/K. ■

It is well known (see, e.g., [18], Corollary 6.33, p. 215) that a maximal torus in a (real) compact analytic group coincides with its centralizer. Below we establish the complex version of this result.

Theorem 5.25 *Let G be a reductive complex analytic group, and let T be a maximal complex torus of G. Then T coincides with its centralizer in G.*

Proof. Let $Z(T)$ denote the centralizer of T in G. We view G as a reductive algebraic group in light of Theorem 5.11. T is then an algebraic subgroup of the algebraic group G, and hence so is its centralizer $Z(G)$. Let X be the maximum compact subgroup of the complex torus T. We prove that X is a maximal compact subgroup of $Z(T)$. Suppose C is any compact subgroup of $Z(T)$ that contains X, and choose a maximal compact subgroup K of G containing C.

Then X is a maximal torus of the real compact group K. In fact, suppose that Y is a torus in K that contains X properly. Then the smallest complex analytic subgroup Y^* that contains Y is a complex torus by Corollary 4.4, and contains the complex torus T properly, contradicting the maximality of T. The centralizer of the maximal torus X in K coincides with X, and this, in particular, shows that $C \subset X$, proving that X is a maximal compact subgroup of $Z(T)$. This also shows that $Z(T)$ is connected. Indeed, $Z(T)$, being an algebraic subgroup, has finitely many connected components, and hence it contains a finite subgroup F so that $Z(T) = FZ(T)_0$. Now FX is a compact subgroup of $Z(T)$. Since X is a maximal compact subgroup of $Z(T)$, $F \subset X$, and hence $F \subset T \subset Z(T)_0$, proving $Z(T) = Z(T)_0$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, respectively. The Lie algebras $\mathcal{L}(T)$ and $\mathfrak{g}$ are the complexification of $\mathcal{L}(X)$ and $\mathfrak{k}$, respectively. Since $\mathcal{L}(X)$ is a Cartan subalgebra of the Lie algebras $\mathfrak{k}$ (see, e.g., [18], Proposition 6.23, p. 210), its complexification $\mathcal{L}(T)$ is a Cartan subalgebra of $\mathfrak{g}$. Noting that the Lie algebra $\mathfrak{g}$ is reductive, $\mathfrak{g}$ has the root space decomposition

$$\mathfrak{g} = \mathcal{L}(T) + \sum_{\alpha \neq 0} \mathfrak{g}_\alpha,$$

where the $\mathfrak{g}_\alpha$ are the (1-dimensional) root spaces for nonzero roots α (see Theorem A.24). Since $Z(T)$ centralizes T, $Ad_G(Z(T))$ leaves each root space stable. This shows that $Ad_G(Z(T))$ is isomorphic with a closed subgroup of $D(n, \mathbb{C})$, where $n = \dim \mathfrak{g}$, and hence it is reductive. On the other hand, if Z denotes the center of G, then the identity component Z_0 is reductive by Corollary 4.23, and Z/Z_0 is finite. Now $Z(T)/Z \cong Ad_G(Z(T))$ is reductive, and hence $Z(T)/Z_0$ is reductive by Lemma 4.33. Consequently, $Z(T)$ is reductive (Theorem 5.24), and we see that its maximal compact group X is full in $Z(T)$ by Theorem 4.31. Since X is already full in T, it follows that $T = Z(T)$, proving our assertion. ■

Chapter 6

Observability in Complex Lie Groups

In this chapter we shall make use of the analytic and algebraic structure of complex analytic groups developed in the earlier chapters to study observability in complex analytic groups. The discussion of observability is carried out in the first two sections for affine and pro-affine algebraic groups, and the remainder of the chapter is devoted to the analytic case ([1]), [7], [11], [24]).

The notion of observability was originally defined for algebraic groups. An algebraic subgroup H of an affine (or more generally, a pro-affine) algebraic group G over a field $\mathbb{F}$ is called *observable* if each rational representation of H extends to a rational representation of the entire group, or to put it another way, if every rational L-module is a sub L-module of a rational G-module. As in the case of algebraic groups, a closed complex Lie subgroup L of a complex analytic group G is said to be *observable* in G if each complex analytic representation of L extends to a complex analytic representation of G.

6.1 Pro-affine Groups and Observability

Throughout this section, we shall assume that $\mathbb{F}$ is an algebraically closed field of characteristic 0, and all affine or pro-affine algebraic groups are assumed to be defined over $\mathbb{F}$. We start with a series of algebraic results which are essential to the subsequent discussions.

Let G be a pro-affine algebraic group, and let $\rho : G \to GL(V, \mathbb{F})$ be a rational representation of G. A *semi-invariant* of G (or simply, *G-semi-invariant*) in V is a nonzero element $v \in V$ spanning a G-stable line in V. In this case, we can write $\rho(x)(v) = \chi(x)v$ for all $x \in G$, where $\chi : G \to \mathbb{F}^*$ is some function. Then the function χ is a rational character of G, which we call the *weight* of v. If G acts on a variety X, we shall also use the term semi-invariant with respect to the action of G on functions on X, which is induced by the action of G on X.

In the following discussion, the concept of the exterior algebra $\wedge(V)$ built on a finite dimensional $\mathbb{F}$-linear space V plays an important role. Here we briefly review this concept. Let n denote the dimension of V. Then $\wedge(V)$ is defined as the quotient algebra of the graded tensor algebra $T(V) = \sum_i T_i(V)$ modulo the ideal generated by the squares of the elements in V, where $T_0(V) = \mathbb{F}$, and, for $i > 0$, $T_i(V) = V \otimes \cdots \otimes V$ (i-times). Thus $\wedge(V)$ is a finite dimensional graded algebra

$$\wedge(V) = \sum_{i=0}^{n} \wedge^i(V),$$

where if $\{v_1, \cdots, v_n\}$ is an ordered basis of V, then the $\begin{pmatrix} n \\ r \end{pmatrix}$ exterior products $v_{i_1} \wedge \cdot \wedge v_{i_r}$ $(i_1 < i_2 < \cdots < i_r)$ form a basis of $\wedge^r(V)$. Note that $\wedge^n(V)$ is 1-dimensional, and $\wedge^r(V) = 0$ for $r > n$. If W is a subspace of V, then $\wedge^r(W)$ is identified with a subspace of $\wedge^r(V)$.

Let G be a group, and let V be a G-module with the corresponding representation $\rho : G \to GL(V, \mathbb{F})$. Then the action of G on V induces an action of G on each homogeneous component $\wedge^i(V)$ of $\wedge(V)$. In particular, the action of $x \in G$ on $\wedge^n(V)$ is the scalar multiplication by the determinant of $\rho(x)$.

Lemma 6.1 *Let V be a finite-dimensional $\mathbb{F}$-linear space, and let W be a d-dimensional subspace of V. Then for $x \in GL(V, \mathbb{F})$, $\wedge^d x\ (\wedge^d(W)) = \wedge^d(W)$ if and only if $x(W) = W$.*

Proof. If $x(W) = W$, then clearly $\wedge^d x\ (\wedge^d(W)) = \wedge^d(W)$. For the converse, choose a basis $v_1, \cdots, v_n$ of V so that $v_1, \cdots, v_d$ is a basis of W and that $v_{m+1}, \cdots, v_{m+d}$ is a basis for $x(W)$, where m is some integer ≥ 0. The element $v_1 \wedge \cdots \wedge v_d$ spans $\wedge^d(W)$, and the element $\wedge^d x\ (v_1 \wedge \cdots \wedge v_d)$ is a scalar multiple of $v_1 \wedge \cdots \wedge v_d$ by hypothesis. On the other hand, $\wedge^d x\ (v_1 \wedge \cdots \wedge v_d)$ is a scalar multiple of $v_{m+1} \wedge \cdots \wedge v_{m+d}$. This shows $m = 0$, proving $x(W) = W$. ∎

Theorem 6.2 *Let H be an algebraic subgroup of an affine algebraic group G. There is a rational representation $\varphi : G \to GL(V, \mathbb{C})$ and a 1-dimensional subspace D of V such that*

$$H = \{x \in G : \varphi(x)(D) = D\}. \qquad (6.1.1)$$

Proof. We set $A = P(G)$. G acts on A by left translations

$$(x, f) \mapsto x \cdot f : G \times A \to A.$$

Let I be the ideal in $P(G)$ of functions vanishing on H. Then I is finitely generated, and we have

$$H = \{x \in G : x \cdot f \in I \ \forall f \in I\}.$$

There is a finite-dimensional left G-stable subspace W of $P(G)$ such that, if $E = W \cap I$, the ideal I is generated by E. We have

$$H = \{x \in G : x \cdot E = E\}. \qquad (6.1.2)$$

In fact, since W and I are both left H-stable, so $x \cdot E \subset E$ for all $x \in H$. Conversely, let $x \in G$ with $x \cdot E \subset E$. Then $x \cdot I = x \cdot (EA) = (x \cdot E)(x \cdot A) = EA = I$, so $x \in H$.

Now let $\rho : G \to GL(W, \mathbb{F})$ denote the representation of G by left translations on W, and put $V = \wedge^d(W)$ and $D = \wedge^d(E)$, where $d = \dim E$. ρ induces a rational representation

$$\varphi = \wedge^d \rho : G \to GL(V, \mathbb{F}).$$

Since $\rho(x)(E) = E$ if and only if $\varphi(x)(\wedge^d(E)) = \wedge^d(E)$ for $x \in G$ by Lemma 6.1, (6.1.1) follows from (6.1.2). ■

Corollary 6.3 *Under the assumption of Theorem 6.2, there exists a rational character χ of H and a finite set $E = \{f_1, \cdots, f_n\}$ of H-semi-invariants in $P(G)$ of the same weight χ such that*

$$H = \{x \in G | \, x \cdot f_i \in \mathbb{F} f_i \text{ for } 1 \leq i \leq n\}.$$

Proof. Let φ be as in Theorem 6.2. Let $e_1, \cdots, e_n$ be a basis of V with $D = \mathbb{F} e_1$, and let $\mu_{i,j}$ denote the (i, j) coordinate function on $\mathfrak{gl}(V, \mathbb{F}) \cong \mathfrak{gl}(n, \mathbb{F})$, the isomorphism being defined relative to the basis above. In this coordinate system, the condition $x \cdot E = E$ (or, equivalently, $x \cdot D = D$) becomes

$$\mu_{i,1} \circ \varphi(x) = 0$$

for all i with $1 < i \leq n$. Define $f_i = \mu_{i,1} \circ \varphi$, for $1 < i \leq n$, and let $\chi = \mu_{1,1} \circ \varphi$. Then χ is a character on H, and (6.1.1) of Theorem 6.2 becomes

$$H = \{x \in G : f_i(x) = 0,\ 1 < i \leq n\}. \tag{6.1.3}$$

For $x \in G$ and $y \in H$, we have

$$\begin{aligned} y \cdot f_i(x) &= f_i(xy) = \mu_{i,1}(\varphi(x)\varphi(y)) \\ &= \sum_k \mu_{i,k} \circ \varphi(x)\mu_{k,1} \circ \varphi(y) \\ &= \mu_{i,1} \circ \varphi(x)\mu_{1,1} \circ \varphi(y) \\ &= \chi(y) f_i(x), \end{aligned}$$

proving that each f_i is an H-semi-invariant of weight χ. Now suppose $x \in G$ such that $x \cdot f_i \in \mathbb{F} f_i$. Then $x \cdot f_i(1)$ is a multiple of $f_i(1)$. Since $f_i(1) = \mu_{i,1} \circ \varphi(1) = 0$, we have $f_i(x) = x \cdot f_i(1) = 0$, and $x \in H$ follows by (6.1.3). ∎

Throughout this chapter, $[R]$, for any integral domain R, denotes the field of fractions of R. Using this notation,

Proposition 6.4 *Let H be an algebraic subgroup of a connected affine algebraic group G. Then*

$$H = \{x \in G : x \cdot f = f \text{ for all } f \in [P(G)]^H\}.$$

Proof. If $x \in H$, then clearly $x \cdot f = f$ for all $f \in [P(G)]^H$. Let $x \in G$ be such that $x \cdot f = f$ for all $f \in [P(G)]^H$, and assume that $x \notin H$. By Theorem 6.2, there is a rational G-module V and a nonzero $v \in V$ such that

$$y \in G \text{ is in } H \text{ if and only if } y \cdot v \in \mathbb{F}v. \tag{6.1.4}$$

Since $x \notin H$ and $x \cdot v \notin \mathbb{F}v$, v and $x \cdot v$ are linearly independent. Choose $\mathbb{F}$-linear functions λ and μ of V such that

$$\lambda(v) = 1,\ \lambda(x \cdot v) = 0;\ \mu(v) = 1,\ \mu(x \cdot v) = 1,$$

and define $g,\ h : G \to \mathbb{F}$ by

$$g(y) = \lambda(y \cdot v),\ h(y) = \mu(y \cdot v)$$

for all $y \in G$. If $\varphi : G \to GL(V, \mathbb{F})$ is the rational representation associated with the G-module V, then g and h are the coefficient functions $g = \varphi_{\lambda,v}$ and $h = \varphi_{\mu,v}$, and hence $g,\ h \in P(G)$. We have $g/h \in [P(G)]^H$, owing to (6.1.4), and since we assume that x fixes every element of $[P(G)]^H$, $x \cdot (g/h) = g/h$, and

$$(g/h)(x) = x \cdot (g/h)(1) = (g/h)(1) = g(1)/h(1) = 1.$$

On the other hand, we have $(g/h)(x) = g(x)/h(x) = 0/1 = 0$. This contradiction shows that $x \in H$, proving our proposition. ∎

The following lemma is an affine (or pro-affine) group version of Lemma 2.17.

Lemma 6.5 *If ρ is a rational representation of an affine algebraic group G on V, then the rational G-module V is embedded as a sub G-module of the direct sum $[\rho] \oplus \cdots \oplus [\rho]$ ($\dim V$ copies).* ∎

Lemma 6.6 *Let H be an algebraic subgroup of a pro-affine algebraic group G. Suppose that, for every 1-dimensional rational H-module that is contained as a sub H-module in a rational G-module, the dual H-module is also a sub H-module of a rational G-module. Then H is observable in G.*

Proof. Let W be a rational H-module and let ρ be the corresponding rational representation of H. W may be identified with a sub H-module of the direct sum of finitely many copies of $[\rho]$ (Lemma 6.5). Since $[\rho] \subset P(H)$ and since $P(G) \to P(H)$ is surjective, W may be written in the form U/V, where U is a sub H-module of a rational G-module M, and where V is a sub H-module of U.

Let $n = \dim_{\mathbb{F}}(V)$, and consider the homogeneous component $\wedge^{n+1}(M)$ of the exterior algebra built on M. The G-module $\wedge^{n+1}(M)$ contains $U \wedge (\wedge^n(V))$ as a sub H-module. Since $V \wedge (\wedge^n(V)) = \wedge^{n+1}(V) = 0$, $U \wedge (\wedge^n(V))$ is isomorphic (as an H-module) with the tensor product $W \otimes \wedge^n(V)$. Let S denote the dual H-module of the 1-dimensional H-module $\wedge^n(V)$. $\wedge^n(V)$ is a 1-dimensional sub H-module of the G-module $\wedge^{n+1}(M)$, and hence it follows from our hypothesis that there is a G-module, T say, that contains S as a sub H-module. Fix nonzero elements $u \in \wedge^n(V)$ and $\lambda \in S$. Then the composite of the $\mathbb{F}$-linear isomorphism

$$w \mapsto w \otimes u \otimes \lambda : W \to W \otimes \wedge^n(V) \otimes S$$

with the injection $W \otimes \wedge^n(V) \otimes S \to M \otimes T$ identifies W with a subspace of $\wedge^{n+1}(M) \otimes T$. Moreover, the action of each $x \in G$ on $\wedge^n(V)$ and on $\wedge^n(V)^\circ$ is the scalar multiplication by $\det(\rho(x))$ and $\det(\rho(x))^{-1}$, respectively, and hence the $\mathbb{F}$-linear isomorphism $w \mapsto w \otimes u \otimes \lambda$ is an isomorphism of H-modules. Thus

$$W \cong W \otimes \wedge^n(V) \otimes S$$

is a sub H-module of the G-module $M \otimes T$, so that Lemma 6.6 is proved. ■

Using the argument used in the proof of Lemma 6.6, we prove

Corollary 6.7 *Every reductive algebraic subgroup of a connected affine algebraic group G is observable in G.*

Proof. Let H be a reductive algebraic subgroup of G, and let W be a rational H-module with the corresponding rational representation ρ. As in the proof of Lemma 6.6, we write $W = U/V$, where U is a sub H-module of a rational G-module M, and where V is a sub H-module of U. Since H is reductive, U contains a sub H-module which is a complement to V. Replacing U by this complement if necessary, we may assume that $V = (0)$, i.e., $W = U$. This shows that the H-module W is embedded into the G-module M as an H-submodule, proving that H is an observable algebraic subgroup of G. ■

Since any H-module and G-module may be constructed from the polynomial algebras $P(H)$ and $P(G)$, respectively (Lemma 6.5), Lemma 6.6 has the following equivalent form:

Lemma 6.8 *Let G be a pro-affine algebraic group over $\mathbb{F}$, and let H be an algebraic subgroup of G. Assume that for any H-semi-invariant $f \in P(G)$ with weight $\gamma : H \to \mathbb{F}^*$, there exists an H-semi-invariant $g \in P(G)$ with weight $\gamma^{-1} : H \to \mathbb{F}^*$. Then H is observable in G.* ■

The following is a characterization of observable algebraic subgroups in a pro-affine group.

Theorem 6.9 *Let G be a connected pro-affine algebraic group over an algebraically closed field $\mathbb{F}$, and let H be an algebraic subgroup of G. Then H is observable in G if and only if* $[P(G)]^H = [P(G)^H]$.

Proof. Assume that H is observable. Clearly $[P(G)]^H \supseteq [P(G)^H]$. To show $[P(G)]^H \subseteq P(G)^H]$, let $q \in [P(G)]^H$. To show $q \in [P(G)^H]$, it is enough to show that

$$(P(G) \cdot q \cap P(G))^H \neq (0).$$

Let V be a nonzero simple rational sub H-module of $P(G) \cdot q \cap P(G)$, and let V^* denote its dual H-module. Since H is observable, V is a sub H-module of some rational G-module W. Since W is isomorphic with a sub G-module of a direct sum of finitely many copies of $P(G)$ by Lemma 6.5, and since V is simple, there is a monomorphism $\phi : V \rightarrow P(G)$. We choose a basis $v_1, \cdots, v_n$ of V such that $v_1(1) \neq 0$ and $v_i(1) = 0$ for $i = 2, \cdots, n$, and let $\lambda_1, \cdots, \lambda_n$ be the basis of V^* which is dual to the basis $v_1, \cdots, v_n$. If we put $g_i = \phi(\lambda_i)$ for $1 \leq i \leq n$, then, for each $x \in G$, the element

$$h = \sum_{i=1}^{n} (g_i \cdot x) v_i$$

belongs to $(P(G) \cdot q \cap P(G))^H$. Now take x such that $g_1(x) \neq 0$. Then

$$h(1) = \sum_{i=1}^{n} (g_i \cdot x)(1) v_i(1) = g_1(x) \neq 0,$$

and hence $h \neq 0$. This proves that $(P(G) \cdot q \cap P(G))^H \neq (0)$.

Now we prove the sufficiency of the condition. Thus we assume $[P(G)]^H = [P(G)^H]$ to show the condition of Lemma 6.8: Assume that, for any H-semi-invariant $f \in P(G)$ with weight $\gamma : H \rightarrow \mathbb{F}^*$, there is an H-semi-invariant $g \in P(G)$ with weight $\gamma^{-1} : H \rightarrow \mathbb{F}^*$. Then H is observable in G. For $y \in H$ and $x \in G$,

$$y \cdot (f \cdot x) = (y \cdot f) \cdot x = \gamma(y) f \cdot x,$$

and hence we have

$$y \cdot (\frac{f \cdot x}{f}) = \frac{y \cdot (f \cdot x)}{y \cdot f} = \frac{\gamma(y) f \cdot x}{\gamma(y) f} = \frac{f \cdot x}{f},$$

proving $\frac{f \cdot x}{f} \in [P(G)]^H$. Since $[P(G)]^H = [P(G)^H]$, there exist $k_x, m_x \in P(G)^H$ such that $\frac{f \cdot x}{f} = \frac{m_x}{k_x}$. Let $x_1, \cdots, x_n \in G$ such that $\{f \cdot x_1, \cdots, f \cdot x_n\}$ is a basis for the finite-dimensional subspace of $P(G)$ that is spanned by set $\{f \cdot x | x \in G\}$. For each $x \in G$,

$$f \cdot x = \sum_{i=1}^{n} a_i(x) f \cdot x_i,$$

where $a_i(x) \in \mathbb{F}$. For each $1 \leq i \leq n$, let $k_i, m_i \in P(G)^H$ such that $\frac{f \cdot x_i}{f} = \frac{m_i}{k_i}$. Since G is connected, $P(G)$ is an integral domain, and since each $k_i \neq 0$, the product $k = k_1 k_2 \dots k_n$ is a nonzero element of $P(G)^H$. We have

$$\begin{aligned} (f \cdot x)k &= \sum_i a_i(x)(f \cdot x_i) k_1 k_2 \dots k_n \\ &= \sum_i a_i(x) f (\frac{m_i}{k_i}) k_1 k_2 \cdots k_n \\ &= \sum_i a_i(x)(m_i \kappa_i') f \end{aligned}$$

where $k_i' = \prod_{j \neq i} k_j$. Let Z be the set of zeros of f in G_B, where B is the fully stable subalgebra generated by the elements k_i, m_i, and f.

For $z \in Z$,

$$(f \cdot x)(z) k(z) = ((f \cdot x)k)(z) = \sum_i a_i(x)(m_i \kappa_i') f(z) = 0$$

and hence $f(zx)k(z) = 0$ for all $x \in G$. Since $f \neq 0$, we have $k(z) = 0$. This proves that the nonzero k vanishes on Z. Since $\mathbb{F}$ is assumed to be algebraically closed, it follows from the Hilbert Nullstellensatz that $k^m \in Bf$. Let $k^m = gf$ for some $g \in B$. Then for $y \in H$, we have

$$y \cdot (gf) = y \cdot k^m = (y \cdot k)^m = k^m = gf.$$

On the other hand,

$$y \cdot (gf) = (y \cdot g)(y \cdot f) = (y \cdot g)\gamma(y) f.$$

Thus $gf = (y \cdot g)\gamma(y) f$, and $(g - (y \cdot g)\gamma(y)) f = 0$. Since $f \neq 0$, $g = (y \cdot g)\gamma(y)$, proving that $y \cdot g = \gamma(y)^{-1} g$, $y \in H$. ∎

6.2 Observability in Algebraic Groups

Some of the equivalent conditions under which a subgroup of an affine algebraic group is observable are discussed in this section. The main result (Theorem 6.13) is preceded by some algebraic preliminaries related to the extension question in ring theory.

For a proof of the following lemma, see, e.g., Corollary 3.3, p. 348, Lang's Algebra (3rd ed.).

Lemma 6.10 *Let R be a subring of a field K, and let $\phi : R \to \mathbb{F}$ be a homomorphism, where $\mathbb{F}$ is an algebraically closed field. Let M be a maximal subring of K such that $M \supseteq R$ and ϕ extends to M. Then M is a valuation ring, i.e., $x \in K, x \neq 0$ implies $x \in M$ or $x^{-1} \in M$.* ∎

As an easy consequence of Lemma 6.10, we have

Corollary 6.11 *Let R and S be subrings of a field K such that $S \supseteq R$, and let $\mathbb{F}$ be an algebraically closed field. If S is integral over R, then every ring homomorphism $\phi : R \to \mathbb{F}$ extends to a ring homomorphism $\phi' : S \to \mathbb{F}$.*

Proof. We shall show that $S \subset M$, where M is as in the lemma above. Let $x \in S$, $x \neq 0$. Then $x \in M$ or $x^{-1} \in M$ by the lemma. Suppose $x^{-1} \in M$. Then $R[x^{-1}] \subset M$. Since x is integral over R, x satisfies an integral equation

$$r_0 + r_1 x + \cdots + r_{n-1}x^{n-1} + x^n = 0$$

with $r_0, \cdots, r_{n-1} \in R$, and this implies

$$x = -r_0(x^{-1})^{n-1} - \cdots - r_{n-1} \in R[x^{-1}],$$

proving $x \in M$. We now have $S \subset M$, and we see that ϕ extends to a homomorphism $\phi' : S \to \mathbb{F}$, proving our assertion. ∎

Proposition 6.12 *Let A be an integral domain and B a subring of A such that A is of finite type over B. Given $a \neq 0$ in A, there exists $b \neq 0$ in B such that any homomorphism ϕ of B into an algebraically closed field $\mathbb{F}$ with $\phi(b) \neq 0$ can be extended to a homomorphism $\phi' : A \to \mathbb{F}$ with $\phi'(a) \neq 0$.*

Proof. Let $x_1, \cdots, x_n \in A$ so that $A = B[x_1, \cdots, x_n]$, and we prove the assertion by induction on n. Assume the assertion in the proposition is true for $n = 1$, and let $B' = B[x_1, \cdots, x_{n-1}]$ so that $A = B'[x_n]$. Thus there exists an element $b' \in B'$, $b' \neq 0$, such that any homomorphism of B' into the algebraically closed field $\mathbb{F}$ not vanishing at b' can be extended to a homomorphism from A to $\mathbb{F}$ not vanishing at a. This together with the induction hypothesis ensures that there exists $b \neq 0$ in B such that any homomorphism ϕ of B into $\mathbb{F}$ with $\phi(b) \neq 0$ extends to a homomorphism $\phi' : A \to \mathbb{F}$ with $\phi'(a) \neq 0$. Thus we may assume without loss of generality that $n = 1$, $A = B[x]$.

Case I. x is transcendental over $[B]$, the field of fractions of B. Write the element a as

$$a = b_0 + b_1 x + \cdots + b_m x^m \qquad (6.2.1)$$

with each $b_i \in B$ and $b_m \neq 0$. Let $\phi : B \to \mathbb{F}$ be a homomorphism with $\phi(b_m) \neq 0$. Then the polynomial

$$\phi(b_0) + \phi(b_1)X + \cdots + \phi(b_m)X^m \in \mathbb{F}[X]$$

is not 0, and hence there exists an element $\alpha \in \mathbb{F}$ such that

$$\phi(b_0) + \phi(b_1)\alpha + \cdots + \phi(b_m)\alpha^m \neq 0. \qquad (6.2.2)$$

The homomorphism ϕ can be extended to a ring homomorphism $\phi' : A = B[x] \to \mathbb{F}$ such that $\phi'(x) = \alpha$. Then (6.2.1) and (6.2.2) ensure that $\phi'(a) \neq 0$, and we see that the assertion is true with $b = b_m$.

Case II. x is algebraic over $[B]$. Then $[B](x)$ is algebraic over $[B]$, and hence $a^{-1} \in [B](x)$ is also algebraic over $[B]$. We may find $v \in B$, $v \neq 0$ such that vx and va^{-1} are both integral over B. In fact, let x and u^{-1} satisfy equations

$$c_0 + c_1 x + \cdots + c_m x^m = 0 \qquad (6.2.3)$$

and

$$d_0 + d_1 a^{-1} + \cdots + d_n a^{-n} = 0 \qquad (6.2.4)$$

with the $c_i, d_j \in B$, $c_m \neq 0, d_n \neq 0$. Multiplying both equations above by suitable elements of B, if necessary, we may assume that $c_m = d_n$. Let $v = c_m (= d_n)$. Multiplying the equations (6.2.3) and

(6.2.4) by v^{m-1} and v^{n-1}, respectively, we obtain integral equations for vx and va^{-1} over B. This shows that x and a^{-1} are integral over $B[v^{-1}]$, and we see that $B[v, vx, va^{-1}]$ is integral over $B[v^{-1}]$. Let $\phi : B \to \mathbb{F}$ be a homomorphism with $\phi(v) \neq 0$. Clearly we can extend ϕ to $B[v^{-1}] \to \mathbb{F}$, and hence to

$$\phi' : B[v, vx, va^{-1}] \to \mathbb{F}$$

by Corollary 6.11. Since $a^{-1} \in B[v, vx, va^{-1}]$, we have $\phi'(a^{-1}) \neq 0$, and hence $\phi'(a) \neq 0$ follows. Since $A = B[x] \subset B[v, vx, va^{-1}]$, ϕ extends to a homomorphism $\phi' : A \to \mathbb{F}$. ■

Theorem 6.13 *Let H be an algebraic subgroup of a connected affine algebraic group G over an algebraically closed field $\mathbb{F}$. The following are equivalent.*

(i) *H is an observable subgroup in G.*

(ii) $[P(G)^H] = [P(G)]^H$.

(iii) *The variety G/H is quasi-affine (i.e., isomorphic with an open subvariety of an affine variety).*

Proof. (i)⇔(ii) follows from Theorem 6.9.

(ii)⇒(iii): Since the field $[P(G)]$ is finitely generated over $\mathbb{F}$, the subfield $[P(G)]^H$ of $[P(G)]$ is finitely generated over $\mathbb{F}$. We therefore deduce from the relation $[P(G)^H] = [P(G)]^H$ that there exists a finitely generated subalgebra A of $P(G)^H$ such that $[A] = [P(G)]^H$. Since $P(G)^H$ is locally finite as a right G-module, we may choose A so that A is stable under the action from the right. Let $V(A)$ denote the affine variety $Hom_{\mathbb{F}-alg}(A, \mathbb{F})$ (see §B.2). The variety $V(A)$ becomes a G-variety, where the (left) action of G on $V(A)$ is the one induced from the action of G on A from the right. Identifying G with $Hom_{\mathbb{F}-alg}(P(G), \mathbb{F})$, the restriction map $\rho : G \to V(A)$ is a morphism of G-varieties (i.e., equivariant with respect to the actions of G on G and on $V(A)$), where the action of the group G on the affine variety G is by left translations. By Proposition 6.12, the image of any principal open set in G under ρ contains a principal open set in the variety $V(A)$, and thus the image $\rho(G)$ contains a nonempty open subset of $V(A)$. Since G acts on $\rho(G)$ transitively, $\rho(G)$ is open in $V(A)$, and hence it is a quasi-affine variety.

Since $[A] = [P(G)]^H$ separates the points of G/H by Proposition 6.4, it follows that A separates the points of G/H. Thus ρ induces an injective morphism $G/H \to V(A)$, and hence we obtain a bijective morphism $\phi : G/H \to \rho(G)$ of G-varieties. We shall show that ϕ is an isomorphism of varieties by showing its inverse ϕ^{-1} is a morphism of varieties. Since $\rho(G)$ is open in $V(A)$, the field $\mathbb{F}(\rho(G))$ of rational functions on $\rho(G)$ coincides with $\mathbb{F}(V(A)) = [A]$. It follows from $[A] = [P(G)]^H$ that the $\mathbb{F}$-algebra morphism $\mathbb{F}(\rho(G)) \to \mathbb{F}(G/H)$ induced by ϕ is an isomorphism, and hence there is a nonempty open set U of $\rho(G)$ such that ϕ induces an isomorphism $\phi^{-1}(U) \to U$ (see, e.g., [19], Proposition 4.7, p. 36). Thus ϕ^{-1} coincides with a morphism on the open set U, and also on the translates $x \cdot U$ for every $x \in G$. Since G acts on $\rho(G)$ transitively, ϕ^{-1} is a morphism. Now we have an isomorphism $\phi : G/H \cong \rho(G)$ of varieties, and G/H is therefore quasi-affine.

(iii)⇒(i): We assume G/H is quasi-affine, and let G/H be an open subvariety of an affine variety, X say. We use Lemma 6.8 to show that H is observable in G. Let $f \in P(G)$ be an H-semi-invariant with weight $\gamma : H \to \mathbb{F}^*$. Let Z be the set of zeros of f in G, and let $\mathcal{I}(Z)$ denote the ideal of $P(G)$ consisting of all polynomial functions vanishing on Z. For $y \in H$ and $z \in Z$, we have

$$f(zy) = (y \cdot f)(z) = \gamma(y)f(z) = 0,$$

proving $ZH = Z$. Let $\pi : G \to G/H$ be the canonical map. From $ZH = Z$, we see that $\pi(G \setminus ZH) \cap \pi(ZH)$ is empty. Since π is an open map, the set $\pi(G \setminus ZH)$ is open, and its complement $\pi(ZH) = \pi(Z)$ is hence closed in G/H. Since $f \neq 0$, $f(x_0) \neq 0$ for some $x_0 \in G$. Then $\pi(x_0) \notin \pi(Z)$, and we may find $h \in P(G)$ such that $h(Z) = 0$, $h(x_0) \neq 0$. In fact, since $\pi(Z)$ is closed in G/H, we have $\pi(Z) = \pi(Z)^* \cap (G/H)$, where $\pi(Z)^*$ denotes the closure of $\pi(Z)$ in X, and hence $\pi(x_0) \notin \pi(Z)^*$. Let $u' \in P(X)$ be such that $u'(\pi(Z)^*) = 0$ and $u'(\pi(x_0)) \neq 0$, and let $u = u'|_{G/H}$. Then we take $h = u \circ \pi$. Since h is constant on the cosets of H, $h \in P(G)^H$, and $h \in \mathcal{I}(Z) = \sqrt{P(G)f}$. By Hilbert Nullstellensatz, $h^n \in P(G)f$ for some positive integer n. Thus $h^n = gf$ for some $g \in P(G)$. For $y \in H$, we have

$$y \cdot (gf) = y \cdot h^n = (y \cdot h)^n = h^n = gf.$$

On the other hand,

$$y \cdot (gf) = (y \cdot g)(y \cdot f) = (y \cdot g)\gamma(y)f.$$

Hence $gf = (y \cdot g)\gamma(y)f$, and $g = \gamma(y)(y \cdot g)$, proving that

$$y \cdot g = \gamma(y)^{-1}g, \ \forall y \in H.$$

This shows that there exists an H-semi-invariant $g \in P(G)$ with weight $\gamma^{-1} : H \to \mathbb{F}^*$, and thus H is observable by Lemma 6.8. ■

Corollary 6.14 *Let G be a connected affine algebraic group. Then every normal algebraic subgroup N of G is observable in G.*

Proof. The variety G/N is affine, and hence the assertion follows from Theorem 6.13. ■

Corollary 6.14 is extended to general (that is, not necessarily connected) affine algebraic groups. For that purpose, we first review the induced representation. For any group G, let $\mathbb{F}[G]$ denote the group algebra of G over a field $\mathbb{F}$. Let H be a subgroup of G such that the index $[G : H] < \infty$, and let W be an H-module with the corresponding representation ρ. H-module W may be viewed as an $\mathbb{F}[H]$-module in a natural way, and hence we form the tensor product

$$\mathbb{F}[G] \otimes_{\mathbb{F}[H]} W,$$

where $\mathbb{F}[G]$ is viewed as an $(\mathbb{F}[G], \mathbb{F}[H])$-bimodule. Thus $\mathbb{F}[G] \otimes_{\mathbb{F}[H]} W$ is given the structure of an $\mathbb{F}[G]$-module, and the canonical injection

$$W \to \mathbb{F}[G] \otimes_{\mathbb{F}[H]} W$$

is a morphism of $\mathbb{F}[H]$-modules. Since $[G : H] < \infty$, we have

$$\dim_{\mathbb{F}} \mathbb{F}[G] \otimes_{\mathbb{F}[H]} W = [G : H] \dim_{\mathbb{F}} W < \infty.$$

View the finite-dimensional $F(G)$-module $\mathbb{F}[G] \otimes_{\mathbb{F}[H]} W$ as a G-module in a natural way, and denote it by $Ind_H^G(W)$. The G-module $Ind_H^G(W)$ is said to be *induced by* W, and the corresponding representation of G is called the representation *induced by* ρ.

As an $\mathbb{F}$-linear space, we have

$$Ind_H^G(W) = \mathbb{F}[G] \otimes_{\mathbb{F}[H]} W = t_1 \otimes W \oplus \cdots \oplus t_m \otimes W,$$

where $t_1, \cdots, t_m$ is a complete set of left coset representatives of H in G.

If H is an algebraic subgroup of an algebraic group G of finite index, and if W is a rational H-module, $Ind_H^G(W)$ is a rational G-module, and this shows that every algebraic subgroup of an affine algebraic group of finite index is observable in G.

Theorem 6.15 *A normal algebraic subgroup of an affine algebraic group is observable.*

Proof. Let N be a normal algebraic subgroup of an affine algebraic group G, and let W be a rational N-module. Let G_0 denote the connected component of G that contains the identity. Since NG_0 is a subgroup of finite index in G, NG_0 is observable in G by the remark preceding Theorem 6.15. Hence our assertion follows as soon as we have shown that N is observable in NG_0. We may therefore assume that $G = NG_0$. We view W as a $(N \cap G_0)$-module. Since the normal subgroup $N \cap G_0$ of G_0 is observable in G_0 (Corollary 6.14), there is a rational G_0-module U and a monomorphism of $(N \cap G_0)$-modules $\phi : W \to U$. Since the subgroup G_0 is of finite index in G, the module $\mathbb{F}[G] \otimes_{\mathbb{F}[G_0]} U$ induced by the G_0-module U is a rational G-module. Choose $x_1, \cdots, x_n \in N$ so that they form a complete set of left coset representatives of G_0 in G, and define

$$\psi : W \to Ind_{G_0}^G(U) = \mathbb{F}[G] \otimes_{\mathbb{F}[G_0]} U$$

by

$$\psi(w) = \sum x_i \otimes \phi(x_i^{-1} w).$$

It is clear that ψ is independent of the particular choice of the coset representatives x_i. Moreover, ψ is a morphism of N-modules. In fact, let $x \in N$. Then $\{x^{-1}x_1, \cdots, x^{-1}x_n\}$ is also a complete set of left coset representatives of G_0 in NG_0, and hence

$$\begin{aligned}
\psi(xw) &= \sum_{i=1}^{n} x_i \otimes \phi(x_i^{-1} x w) \\
&= x \sum_{i=1}^{n} (x^{-1}x_i) \otimes \phi((x^{-1}x_i)^{-1} w) \\
&= x\psi(w).
\end{aligned}$$

Since ψ is injective, the N-module W is embedded into the rational G-module $Ind_{G_0}^G(U)$ as a sub N-module, proving that N is observable in G. ∎

6.3 Extension of Representative Functions

Let G be a faithfully representable complex analytic group, and let $\tau : G \to A(G)$ be the canonical map from G into the universal algebraic hull $A(G)$ (see §2.5). Since G is faithfully representable, τ is injective, and in this case, we sometimes identify the elements of G with their images in $A(G)$ under τ. Below (Theorem 6.17) we first obtain the analytic version of Theorem 6.9. For that, we need

Lemma 6.16 *Let G be a faithfully representable complex analytic group and let L be a closed complex analytic subgroup of G. Let L^* denote the Zariski closure of L in the pro-affine algebraic group $A(G)$. If L is observable in G, then L^* is observable in $A(G)$. Conversely, if L^* is observable in $A(G)$, and if the restriction map $R(G) \to R(L)$ is surjective, then L is observable in G.*

Proof. Assume that L is observable in G, and let V be a rational L^*-module. Viewed as an analytic L-module, V is a sub L-module of an analytic G-module W, and W is, in turn, a rational $A(G)$-module (Proposition 2.21). Thus the L^*-module V is a sub L^*-module of the $A(G)$-module W, proving that L^* is observable in $A(G)$.

Assume that L^* is observable in $A(G)$ and that the restriction map $R(G) \to R(L)$ is surjective. We show that L is observable in G. We first note that since the restriction map $R(G) \to R(L)$ is surjective, the morphism

$$A(L) = Hom_{\mathbb{C}-alg}(R(L), \mathbb{C}) \to A(G) = Hom_{\mathbb{C}-alg}(R(G), \mathbb{C})$$

is an injection, and maps $A(L)$ onto L^*. We may therefore identify L^* with the universal algebraic hull $A(L)$ of L. Now let V be a complex analytic L-module. Then V is a rational L^*-module again by Proposition 2.21, and since L^* is observable in $A(G)$, V is a sub L^*-module of a rational $A(G)$-module W. Thus the analytic L-module V is then a sub L-module of the analytic G-module W, proving that L is observable in G. ∎

Theorem 6.17 *Let L be a closed complex analytic subgroup of a faithfully representable complex analytic group G. Then the following are equivalent.*

(i) *L is observable in G.*

(ii) *$[R(G)]^L = [R(G)^L]$, and the restriction map $R(G) \to R(L)$ is surjective.*

Proof. Put $A = A(G)$, and let L^* denote the Zariski closure of L in A. Then $P(A) = R(G)$, and we have

$$R(G)^{L^*} = R(G)^L; \ [R(G)]^{L^*} = [R(G)]^L. \qquad (6.3.1)$$

(i)⇒(ii): Assume that L is observable in G. Since the algebraic subgroup L^* is observable in A by Lemma 6.16, we have $[P(A)^{L^*}] = [P(A)]^{L^*}$ (Theorem 6.9), and $[R(G)]^L = [R(G)^L]$ follows from (6.3.1). The restriction map $R(G) \to R(L)$ is surjective by Lemma 3.5.

(ii)⇒(i): Let $[R(G)]^L = [R(G)^L]$, and suppose the restriction map $R(G) \to R(L)$ is surjective. Then $[R(G)]^{L^*} = [R(G)^{L^*}]$ by (6.3.1), and this implies that the algebraic subgroup L^* is observable in A by Theorem 6.9. Thus L is observable in G by Lemma 6.16. ■

In light of Theorem 6.17, we shall examine below the extension property of representative functions of analytic subgroups. Recall that every faithfully representable complex analytic group G has the maximum algebraic subgroup, which we denote by $\mathcal{M}(G)$ (Theorem 5.16).

Theorem 6.18 *Let L be a closed complex analytic subgroup of a faithfully representable complex analytic group G. Then the following are equivalent.*

(i) *The restriction map $R(G) \to R(L)$ is surjective.*

(ii) *$\mathcal{M}(L) = L \cap \mathcal{M}(G)$, and $\mathcal{M}(L)$ is an algebraic subgroup of $\mathcal{M}(G)$.*

Proof. Let $N = N(G)$, $V = N(L)$, and choose maximal reductive complex analytic subgroups H and D of G and L, respectively, so that $\mathcal{M}(G) = HN$ and $\mathcal{M}(L) = DV$ (Theorem 5.16). The condition (ii) in the theorem then becomes $L \cap HN = DV$. We put $M = L \cap HN$.

Assume that the restriction map $R(G) \to R(L)$ is surjective. DV is an algebraic subgroup of G by Proposition 5.19, and hence $DV \subset HN \cap L = M$. To prove $DV \supset M$, we first note that M is topologically connected. Indeed, the canonical map $L/M_0 \to L/M$, where M_0 denotes the identity component of M, is a covering map of L/M. On the other hand, the quotient group LHN/HN is a closed complex vector subgroup of G/HN as an analytic subgroup of the vector group G/HN, and $L/M \cong LHN/HN$. From this, we see that L/M is a vector group, and the covering morphism $L/M_0 \to L/M$ is therefore an isomorphism, proving $M = M_0$. Since M is a closed normal subgroup of L such that L/M is a complex vector group, the restriction map $R(L) = R(L, V) \to R(M, V)$ is surjective by Lemma 5.4, and composing this with the surjection $R(G) \to R(L)$, we see that $R(G) \to R(M, V)$ is surjective. This, in particular, shows that $R(HN, N) \to R(M, V)$ is surjective. $R(HN, N)$ is the polynomial algebra of the algebraic group HN (see Remark 5.17), and hence it is finitely generated. However, by Corollary 5.7, $R(M)$ is *not* finitely generated, unless the vector group M/DV is trivial. Hence we must have $M = DV$, proving (ii).

Now assume (ii), i.e., $M = DV$. Since LHN is a closed normal complex analytic subgroup of G such that G/LHN is a vector group, the restriction map

$$R(G) \to R(LHN, N)$$

is surjective by Lemma 5.4. To show that the map $R(G) \to R(L)$ is surjective, it is therefore enough to show that

$$R(LHN, N) \to R(L)$$

is surjective. We have an isomorphism $L/DV = L/M \cong LHN/HN$ of vector groups, and hence we apply Corollary 5.6 to find complex one-parameter subgroups $Q_1, \cdots, Q_s$ of L such that LHN is written in successive semidirect products

$$LHN = Q_s \cdots Q_1 \cdot (HN).$$

Then L is also expressed in successive semidirect products

$$L = Q_s \cdots Q_1 \cdot (DV),$$

and $R(L)$ and $R(LHN, N)$ may be expressed as

$$R(L) = R(Q_s) \otimes \cdot \cdot \otimes R(Q_1) \otimes R(DV, V); \qquad (6.3.2)$$
$$R(LHN, N) = R(Q_s) \otimes \cdot \cdot \otimes R(Q_1) \otimes R(HN, N). \quad (6.3.3)$$

The groups DV and HN are algebraic subgroups of the analytic groups L and G, respectively, in the sense of §5.1 and we have $P(DV) = R(DV, V)$ and $P(HN) = R(HN, N)$ (Remark 5.17). VD is an algebraic subgroup of the algebraic group HN by Proposition 5.19, and hence the restriction map

$$P(HN) \to P(DV)$$

is surjective. Thus the canonical map

$$R(Q_s) \otimes \cdot \cdot \otimes R(Q_1) \otimes R(HN, N) \to R(Q_s) \otimes \cdot \cdot \otimes R(Q_1) \otimes R(DV, V)$$

is surjective, and hence the map $R(LHN, N) \to R(L)$ is surjective by (6.3.2) and (6.3.3). ∎

6.4 Structure of Observable Subgroups

In this section we state and prove the main result (Theorem 6.20) on observable analytic subgroups.

Rational Representations of Linear Lie Groups Let B be a closed complex Lie subgroup of a linear complex algebraic group Q, and let B^* denote the Zariski closure of B in Q. A complex analytic representation $\rho : B \to GL(V, \mathbb{C})$ is called *rational* if it is the restriction to B of a rational representation $\varphi : K \to GL(V, \mathbb{C})$ of some algebraic subgroup K of Q that contains B. In this case, ρ extends uniquely to a rational representation $\rho^* : B^* \to GL(V, \mathbb{C})$. The restriction map $P(B^*) \to R(B)$ is an injection, and if S denotes the image of $P(B^*)$ under this injection, S is a fully stable subalgebra of $R(B)$. From now on we identify $P(B^*)$ with S, i.e., we view $P(B^*)$ as a fully stable subalgebra of $R(B)$. Now let B be a closed complex analytic subgroup of a complex linear algebraic group Q, and let ρ be a complex analytic representation of B.

Lemma 6.19 *ρ is rational if and only if $[\rho] \subset P(B^*)$.*

Proof. Assume ρ is rational, and let ρ^* be a rational representation of B^* such that $\rho^*|_B = \rho$. Then we have $[\rho] = [\rho^*] \subset P(B^*)$.

Conversely, assume $[\rho] \subset P(B^*)$. We are to show that ρ extends to a rational representation $B^* \to GL(V, \mathbb{C})$. The bistable subspace $[\rho]$ of $R(B)$ is also stable under $A(S) = A(P(B^*)) = B^*$ by Lemma 2.11, and this makes $[\rho]$ a rational B^*-module. By Lemma 2.17, the B-module V may be viewed as a sub B-module of a finite direct sum

$$W = [\rho] \oplus \cdots \oplus [\rho]$$

of the rational B^*-module $[\rho]$. Our assertion is equivalent to showing that V is B^*-stable. Choose a basis $v_1, v_2, \cdots, v_n$ of the linear space V, and extend this to a basis $v_1, v_2, \cdots, v_m$ $(m \geq n)$ of the linear space W. For each $y \in B^*$, we write

$$y \cdot v_i = a_{1i}(y)v_1 + \cdots + a_{mi}(y)v_m,$$

where $a_{ki}(y) \in \mathbb{C}$. W is a rational Q-module, and hence the maps $y \mapsto a_{ki}(y)$ $(1 \leq k, i \leq m)$ are all rational functions on Q. Since V is B-stable, $a_{ki} = 0$ on B (and hence also on B^*) for all k, i with $n+1 \leq k \leq m$, $1 \leq i \leq n$. This shows that V is B^*-stable, and proof of our lemma is complete. ∎

We now state our main result on the observability of analytic subgroups of faithfully representable groups.

Theorem 6.20 *Let G be a faithfully representable complex analytic group and let L be a closed complex analytic subgroup of G. Then L is observable in G if and only if the following conditions are satisfied.*

(i) $\mathcal{M}(L) = L \cap \mathcal{M}(G)$, *and* $\mathcal{M}(L)$ *is an algebraic subgroup of* $\mathcal{M}(G)$, *and*

(ii) $\mathcal{M}(L)$ *is observable in* $\mathcal{M}(G)$ *(in the category of affine algebraic groups).*

Proof. Let N denote the representation radical of G, and let H be a maximal reductive subgroup of G, so that $\mathcal{M}(G) = HN$.

Suppose that L is observable in G. Then we have the surjective restriction map $R(G) \to R(L)$ (Lemma 3.5), and by Theorem 6.18, $\mathcal{M}(L) = L \cap \mathcal{M}(G)$ and $\mathcal{M}(L)$ is an algebraic subgroup of $\mathcal{M}(G)$, proving (i). Now we prove (ii): $L \cap \mathcal{M}(G)$ is observable in $\mathcal{M}(G)$. Let ρ be a rational representation of $L \cap \mathcal{M}(G)$. Since G is faithfully representable, G may be viewed as a subgroup of a linear algebraic group. Let L^* denote the Zariski closure of L in this algebraic group. The normal subgroup $\mathcal{M}(L) = L \cap \mathcal{M}(G)$ of L is an algebraic subgroup of L^* by Theorem 5.16, and hence it is observable in L^* by Theorem 6.15. Thus ρ extends to a rational representation of L^*, and the restriction σ of this representation to L is an extension of ρ. Now L is observable in G, and hence σ has an extension to a complex analytic representation, $\widetilde{\sigma}$ say, of G. The restriction of $\widetilde{\sigma}$ to $\mathcal{M}(G)$ is a rational representation of $\mathcal{M}(G)$ (Proposition 5.1), and it is an extension of ρ, proving that $L \cap \mathcal{M}(G)$ is observable in $\mathcal{M}(G)$.

We now prove that the conditions (i) and (ii) of the theorem are sufficient. Thus assume that L satisfies the conditions (i) and (ii), and set $M = L \cap \mathcal{M}(G) = L \cap HN$. The subgroup LHN/HN is a closed (vector) subgroup of G/HN as a complex analytic subgroup of the vector group G/HN, and hence LHN is a closed normal subgroup of G and G/LHN is a complex vector group. By Lemma 5.4, every N-unipotent complex analytic representation of LHN is extendable to a complex analytic representation of G. Therefore the observability of L in G follows as soon as we have shown the following:

Proposition 6.21 *Any complex analytic representation of L extends to an N-unipotent complex analytic representation of LHN.*

Our proof of Proposition 6.21 is lengthy and is divided into several parts.

(**A**) We show that the restriction map $R(LHN, N) \to R(HN, N)$ induces an isomorphism:

$$R(LHN, N)^L \cong P(HN)^M. \qquad (6.4.1)$$

In fact, $f \in R(LHN, N)^L$ implies $f|_{HN} \in P(HN)^M$ (note that $P(HN) = R(HN, N)$), and the map $f \mapsto f|_{HN}$ is an injection. To show it is surjective, let $g \in P(HN)^M$, and choose closed complex 1-parameter subgroups $P_1, \cdots, P_r$ of L such that L is expressed in successive semidirect products $L = P_r \cdots P_1 M$ of the normal subgroup M with the subgroups $P_1, \cdots, P_r$. Then LHN is written in successive semidirect products $LHN = P_r \cdots P_1 \cdot HN$. For $1 \le i \le r$, we define

$$g_i : P_i \cdots P_1 \cdot HN \to \mathbb{C}$$

by $g_i(a_i \cdots a_1 z) = g(z)$ for $a_j \in P_j$ $(1 \le j \le i)$ and $z \in HN$. We show that

$$g_i \in R(P_i \cdots P_1 \cdot HN, N)^{P_i \cdots P_1 M} \tag{6.4.2}$$

for all i. For simplicity, we let

$$B_0 = HN;\ A_0 = M;\ B_i = P_i \cdot \cdot P_1 \cdot HN;\ A_i = P_i \cdot \cdot P_1 \cdot M$$

for $1 \le i \le r$. Then $B_{i+1} = P_{i+1} \cdot B_i$ (semidirect product). Assume $g_i \in R(B_i, N)^{A_i}$, and we want to show $g_{i+1} \in R(B_{i+1}, N)^{A_{i+1}}$. Note $g_{i+1} = g_i^+$, where $g_i^+(xb) = g_i(b)$, $x \in P_{i+1}$ and $b \in B_i^+$ as in Lemma 3.2. Thus our assertion follows from Lemma 3.2 as soon as we have shown that the space spanned by $g_i \circ \kappa(P_{i+1})$ is finite-dimensional, where, for $a \in P_{i+1}$, $\kappa(a)$ denotes the automorphism of B_i given by $b \mapsto aba^{-1}, b \in B_i$. We have $g_i \circ \kappa(P_{i+1}) \subset R(B_i, N)$ by Lemma 3.1, and since P_{i+1} normalizes A_i, the identity

$$((aya^{-1}) \cdot g_i) \circ \kappa(a) = y \cdot (g_i \circ \kappa(a)) \text{ for } y \in A_i \text{ and } a \in P_{i+1}$$

(see (3.2.2) in the proof of Lemma 3.1) implies

$$g_i \circ \kappa(P_{i+1}) \subset R(B_i, N)^{A_i}.$$

Now the restriction map

$$R(B_i, N)^{A_i} \to P(HN)^M$$

is injective, and therefore to show that the subspace of $R(B_i, N)^{A_i}$ spanned by $g_i \circ \kappa(P_{i+1})$ is finite-dimensional, it is enough to show that its image

$$g \circ \kappa(P_{i+1})|_{HN}$$

spans a finite-dimensional subspace of $P(HN)^M$. Since the map

$$R(G) \to P(HN)$$

is surjective by Corollary 5.5, we can find an element $\overline{g} \in R(G)$ with $\overline{g}|_{HN} = g$. Now we have

$$g \circ \kappa(a)|_{HN} = (a^{-1} \cdot \overline{g} \cdot a)|_{HN}$$

for $a \in P_{i+1}$. As $\{a^{-1} \cdot \overline{g} \cdot a : a \in P_{i+1}\}$ spans a finite-dimensional space, it follows that $g \circ \kappa(P_{i+1})|_{HN}$ spans a finite-dimensional space. We now have $g_{i+1} \in R(B_{i+1}, N)^{A_{i+1}}$, and (6.4.2) is obtained by using induction starting with $g_0 = g$. Now we have

$$g_r \in R(B_r, N)^{A_r} = R(LHN, N)^L$$

and $g_r|_{HN} = g$, establishing that the restriction map is surjective.

(**B**) By the condition (ii), the algebraic subgroup M is observable in the linear algebraic group HN. Then $P(HN)^M$ separates the points of the quotient space HN/M.

To see this, let $x \in HN$ such that $xM \neq M$. Since $x \notin M$, and since M is an algebraic subgroup of HN, there exists $g \in [P(HN)]^M$ such that $xg \neq g$ by Proposition 6.4. Since M is observable in HN, $[P(HN)^M] = [P(HN)]^M$ by Theorem 6.13, and hence there exists $f \in P(HN)^M$ such that $x \cdot f \neq f$. Let $y \in HN$ such that $x \cdot f(y) \neq f(y)$. Then $(f \cdot y)(x) \neq (f \cdot y)(1)$, and $f \cdot y \in P(HN)^M$ implies that $f \cdot y$ is constant on each coset of M in HN. This shows that $(f \cdot y)(xM) \neq (f \cdot y)(M)$, proving that $P(HN)^M$ separates the points of HN/M.

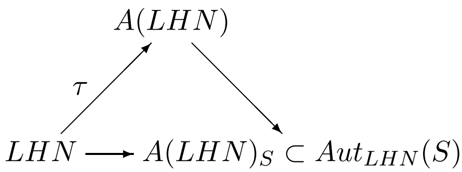
(**C**) We next show that the algebra $R(LHN, N)$ contains a fully stable finitely generated subalgebra S such that

(a) S separates the points of LHN, and

(b) S^L separates the points of LHN/L.

In fact, the field $[P(HN)]$ of fractions of $P(HN)$ is finitely generated over $\mathbb{C}$, and hence the subfield $[P(HN)]^M (= [P(HN)^M])$ is finitely generated. There is a finitely generated subalgebra A of $P(HN)^M$ such that $[A] = [P(HN)]^M$. Since $P(HN)^M$ separates the points of HN/M, so does $[P(HN)]^M = [A]$, and hence A separates the points of HN/M. Consequently, any finite subset, say $\{f_1, \cdots, f_m\}$, which generates the $\mathbb{C}$-algebra A, must separate the points of HN/M. Let $g_i \in R(LHN, N)^L$ correspond to f_i, for $1 \leq i \leq m$, under the isomorphism (6.4.1) of (A). Then $g_1, \cdots, g_m$ separate the points of the coset space LHN/L. Note that $R(G, N)$ contains a finite subset that separates the points of G, for example, a basis of $[\psi]$, where ψ is any faithful complex analytic representation of G. Since the restriction map $R(G, N) \to R(LHN, N)$ is surjective by Lemma 5.4, we may find elements $h_1, \cdots, h_k \in R(LHN, N)$ which separate the elements of LHN. Now we take S to be the fully stable subalgebra of $R(LHN, N)$ generated by the elements $h_1, \cdots, h_k;\ g_1, \cdots, g_m$.

(**D**) Returning now to the proof of Proposition 6.21, choose a finitely generated fully stable subalgebra S of $R(LHN, N)$ satisfying the conditions in (C), and recall that $Aut_{LHN}(S)$ denote the pro-affine algebraic group consisting of all proper automorphisms of S (see §2.3). Since S separates the points of LHN, the canonical map $LHN \to Aut_{LHN}(S)$ is an embedding, and, for any subgroup K of LHN, the image of K under the canonical map is denoted by K_S. Since HN is the maximum algebraic subgroup of G, $(HN)_S = H_S N_S$ is Zariski closed in $Aut_{LHN}(S)$ (Theorem 5.16). We have a commutative diagram



On the other hand, the restriction image $A(LHN)_S$ of the pro-affine algebraic group $A(LHN)$ is an algebraic subgroup of $Aut_{LHN}(S)$, and hence

$$A(LHN)_S = (L_S H_S N_S)^* = L_S^* H_S N_S.$$

Clearly $L_S^* \subset (L_S^* H_S N_S)^{S^L}$, and hence

$$L_S \subset L_S^* \cap L_S H_S N_S \subset (L_S^* H_S N_S)^{S^L} \cap L_S H_S N_S = L_S,$$

where the equality holds because S^L separates the points of LHN/L, proving

$$L_S H_S N_S \cap L_S^* = L_S.$$

We then have

$$L_S^* \cap H_S N_S = L_S \cap H_S N_S, \tag{6.4.3}$$

and this equality establishes an isomorphism (of algebraic varieties)

$$L_S^* H_S N_S / L_S^* \cong H_S N_S / (L_S^* \cap H_S N_S) \cong HN/M.$$

Since M is assumed to be observable in HN, HN/M is quasi-affine by Theorem 6.13. Consequently, $L_S^* H_S N_S / L_S^*$ is quasi-affine, and the algebraic subgroup L_S^* is observable in the algebraic group $L_S^* H_S N_S$

again by Theorem 6.13. On the other hand, the identity (6.4.3) also implies that the injection $L \to L_S^*$ induces an isomorphism

$$\phi_S : LHN/L \to L_S^* H_S N_S / L_S^* \tag{6.4.4}$$

of complex analytic manifolds.

(**E**) With this preparation, we are ready to prove Proposition 6.21. Thus let ρ be a complex analytic representation of L, and we want to show that ρ is extendable to an N-unipotent analytic representation of LHN. Let S be as in (C), and first assume $[\rho] \subset S_L (= P(L_S^*))$. For the sake of simplicity of the notation, we identify LHN with its image $L_S H_S N_S$ in $A(LHN)_S$. By Lemma 6.19, ρ is rational, and hence there is rational representation ρ^* of L^* such that $\rho^*|_L = \rho$. As we have seen above, L^* is observable in L^*HN, and thus ρ^* extends to a rational representation $\overline{\rho}$ of L^*HN. The restriction of $\overline{\rho}$ to LHN is a desired extension of ρ.

Next assume that $[\rho] \not\subset S_L$. Since the restriction map

$$R(G) \to R(L) = R(L, N(L))$$

is surjective, so is the restriction map $R(LHN, N) \to R(L, N(L))$. This enables us to choose a finitely generated fully stable subalgebra T of $R(LHN, N)$ that contains S such that $[\rho] \subset T_L$. Then we have a commutative diagram

$$\begin{array}{ccc} LHN/L & \xrightarrow{\phi_T} & L_T^* H_T N_T / L_T^* \\ & {\scriptstyle \phi_S} \searrow \quad \swarrow & \\ & L_S^* H_S N_S / L_S^* & \end{array}$$

Since ϕ_S is an isomorphism of analytic manifolds (6.4.4), it follows in particular that ϕ_T is injection, and

$$L_T^* \cap H_T N_T = L_T$$

follows. Thus we have an isomorphism of analytic manifolds:

$$L_T^* H_T N_T / L_T^* \cong H_S N_S / (L_T^* \cap H_T N_T) \cong HN/M.$$

Moreover, T^L separates the points of LHN/L, and $[\rho] \subset T_L$. Now we replace S by the fully stable subalgebra T and argue as before to get a desired analytic extension of ρ to LHN. ■

Corollary 6.22 *Every reductive analytic subgroup L of a faithfully representable complex analytic group G is observable.*

Proof. Since $\mathcal{M}(L) = L$, and $L \subset \mathcal{M}(G)$, and since any reductive algebraic subgroup of an algebraic group is observable (Corollary 6.7), the assertion follows from Theorem 6.18 and Theorem 6.20. ∎

Appendix A

Elementary Theory of Lie Algebras

For reference we assemble here some basic concepts and results, mostly without proof, from the theory of Lie algebras that are needed for our exposition ([2], [20], [21]).

Throughout this section we shall assume that all Lie algebras are finite-dimensional and over a field $\mathbb{F}$ of characteristic 0, unless stated otherwise.

A.1 Preliminaries

Basic Definitions Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. For subsets A and B of $\mathfrak{g}$, let $[A, B]$ denote the linear span (over $\mathbb{F}$) of the set $\{(a, b) : a \in A,\ b \in B\}$. $\mathfrak{g}$ is called *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$. An $\mathbb{F}$-linear subspace $\mathfrak{a}$ of $\mathfrak{g}$ is called an *ideal* of $\mathfrak{g}$ if $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$ together imply $[x, y] \in \mathfrak{a}$. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{g}$, then so are $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a}, \mathfrak{b}]$. In particular, $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, called the *derived ideal* of $\mathfrak{g}$, and is often denoted by $\mathfrak{g}'$. A nonabelian Lie algebra $\mathfrak{g}$ is called *simple* if it has no ideals except itself and 0.

For a subset X of $\mathfrak{g}$, the set

$$Z_{\mathfrak{g}}(X) = \{y \in \mathfrak{g} : [y, x] = 0 \text{ for all } x \in X\}$$

is a Lie subalgebra of $\mathfrak{g}$; we call it the *centralizer* of X in $\mathfrak{g}$. We call $Z_{\mathfrak{g}}(\mathfrak{g})$ the *center* of $\mathfrak{g}$ and denote it by $Z(\mathfrak{g})$.

Also for a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the set

$$N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} : [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

is a subalgebra of $\mathfrak{g}$, called the *normalizer of* $\mathfrak{h}$ *in* $\mathfrak{g}$.

Given a finite-dimensional $\mathbb{F}$-linear space V, the linear space $End_{\mathbb{F}}(V)$ is made into a Lie algebra over $\mathbb{F}$ if we define the $[\ ,\ ]$ operation by $[A, B] = A \circ B - B \circ A$ for all $A, B \in End_{\mathbb{F}}(V)$. This Lie algebra is called the *general linear Lie algebra* of V, and we denote it by $\mathfrak{gl}(V, \mathbb{F})$. The $\mathbb{F}$-algebra $Mat_n(\mathbb{F})$ of all $n \times n$ matrices over $\mathbb{F}$ is, similarly, made into a Lie algebra $\mathfrak{gl}(n, \mathbb{F})$.

An $\mathbb{F}$-linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* of $\mathfrak{g}$ if it satisfies

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in \mathfrak{g}$. The set $Der(\mathfrak{g})$ of all derivations of $\mathfrak{g}$ is a subalgebra of the general linear Lie algebra $\mathfrak{gl}(\mathfrak{g}, \mathbb{F})$.

Semidirect Sum We define a semidirect sum of two Lie algebras. Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras over a field $\mathbb{F}$, and suppose $\rho : \mathfrak{b} \to Der(\mathfrak{a})$ is a morphism of Lie algebras. The $\mathbb{F}$-linear space $\mathfrak{a} \oplus \mathfrak{b}$ is made into a Lie algebra by defining the bracket operation $[\ ,\ ]$ by

$$[(a_1, b_1), (a_2, b_2)] = ([a_1, a_2] + \rho(b_1)(a_2) - \rho(b_2)(a_1), [b_1, b_2])$$

for $(a_1, b_1),\ (a_2, b_2) \in \mathfrak{a} \oplus \mathfrak{b}$. The Lie algebra hence obtained is called the *semidirect sum* of $\mathfrak{a}$ by $\mathfrak{b}$ with respect to ρ, and is denoted by $\mathfrak{a} \oplus_\rho \mathfrak{b}$.

Modules over Lie Algebras Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. An $\mathbb{F}$-linear space E is said to be a *module* over $\mathfrak{g}$ (or simply a $\mathfrak{g}$-module) if there is an $\mathbb{F}$-bilinear map

$$(x, e) \mapsto x \cdot e : L \times E \to E$$

such that

$$[x, y] \cdot e = x \cdot (y \cdot e) - y \cdot (x \cdot e)$$

for all $x, y \in \mathfrak{g}$ and $e \in E$. Given a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, $\mathfrak{g}$-modules form a category in which morphisms are $\mathbb{F}$-linear maps commuting with the action of $\mathfrak{g}$.

Let V be a finite-dimensional $\mathbb{F}$-linear space. Any representation ρ of $\mathfrak{g}$ on V, i.e., any morphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \mathbb{F})$ of Lie algebras, induces a $\mathfrak{g}$-module structure on V, which is given by

$$(x, v) \mapsto x \cdot v = \rho(x)(v) : \mathfrak{g} \times V \to V,$$

and, conversely, a $\mathfrak{g}$-module structure on V determines a representation of $\mathfrak{g}$ on V in an obvious way.

For $x \in \mathfrak{g}$, the map $y \mapsto [x, y] : \mathfrak{g} \to \mathfrak{g}$ is a derivation of $\mathfrak{g}$, which we denote by $ad_{\mathfrak{g}}(x)$. Then $x \mapsto ad_{\mathfrak{g}}(x)$ defines a representation $ad_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}, \mathbb{F})$, which we call the *adjoint representation* of $\mathfrak{g}$.

Invariant Bilinear Form Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. An *invariant bilinear form* on $\mathfrak{g}$ is an $\mathbb{F}$-bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ such that

$$\beta([x, y], z) + \beta(y, [x, z]) = 0$$

for all $x, y, z \in \mathfrak{g}$. Suppose V is a finite-dimensional $\mathfrak{g}$-module with the corresponding representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \mathbb{F})$. The bilinear form

$$\beta_\rho : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F},$$

defined by

$$\beta_\rho(x, y) = Tr(\rho(x) \circ \rho(y)),$$

for $x, y \in \mathfrak{g}$ is called the *trace form* on $\mathfrak{g}$ associated with ρ. It is easy to show that β_ρ is a symmetric invariant $\mathbb{F}$-bilinear form on $\mathfrak{g}$. The trace form associated with the adjoint representation of a Lie algebra $\mathfrak{g}$ is called the *Killing form* on $\mathfrak{g}$. Thus the Killing form on $\mathfrak{g}$ is the bilinear form $\kappa_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ given by

$$\kappa_{\mathfrak{g}}(x, y) = Tr(ad(x) \circ ad(y))$$

for $x, y \in \mathfrak{g}$, and it satisfies the invariance condition

$$\kappa_{\mathfrak{g}}([x, y], z) + \kappa_{\mathfrak{g}}(y, [x, z]) = 0 \qquad \text{(A.1.1)}$$

for all $x, y, z \in \mathfrak{g}$.

Lemma A.1 *If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\kappa_{\mathfrak{h}}$ is the restriction of $\kappa_{\mathfrak{g}}$.* ■

A.2 Nilpotent and Solvable Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. We define a sequence of ideals $\mathfrak{g}^k$ of $\mathfrak{g}$:

$$\mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^k \supseteq \cdots$$

by $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$ for $k = 1, 2, \cdots$. This sequence is called the *derived series* of $\mathfrak{g}$. $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^n = 0$ for some integer $n > 0$. Let $\mathfrak{t}(n, \mathbb{F})$ denote the Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ consisting of all upper triangular matrices. Then $\mathfrak{t}(n, \mathbb{F})$ is solvable.

We collect a few properties of solvability as follows.

Proposition A.2 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$.*

(i) *If $\mathfrak{g}$ is solvable, so are all subalgebras and homomorphic images of $\mathfrak{g}$.*

(ii) *If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}$ such that $L/\mathfrak{a}$ is solvable, then $\mathfrak{g}$ is solvable.*

(iii) *If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals of $\mathfrak{g}$, then so is $\mathfrak{a} + \mathfrak{b}$.* ■

As an application of Proposition A.2, we can show that $\mathfrak{g}$ contains a unique maximal solvable ideal R in the sense that it contains all solvable ideals. In fact, by finite dimensionality of $\mathfrak{g}$, $\mathfrak{g}$ contains a maximal solvable ideal R (i.e., a solvable ideal not contained in any larger solvable ideal), and if $\mathfrak{a}$ is any other solvable ideal, then $\mathfrak{a} + R = R$ by maximality, or $\mathfrak{a} \subseteq R$. We call R the *radical* of $\mathfrak{g}$, and denote it by $Rad(\mathfrak{g})$. If $\mathfrak{g} \neq 0$ and $Rad(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is called *semisimple*. Clearly a simple Lie algebra is semisimple, and if $\mathfrak{g}$ is not solvable, then $\mathfrak{g}/Rad(\mathfrak{g})$ is semisimple.

Define a sequence of ideals $\mathfrak{g}_k$ of $\mathfrak{g}$:

$$\mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq \cdot\cdot \supseteq \mathfrak{g}_k \supseteq \cdots$$

by $\mathfrak{g}_0 = \mathfrak{g}$ and $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$, $k = 1, 2, \cdots$. This sequence of ideals is called the *descending (or lower) central series* of $\mathfrak{g}$. If $\mathfrak{g}_k = 0$ for some integer $k > 0$, $\mathfrak{g}$ is called a *nilpotent Lie algebra* . Since $\mathfrak{g}^k \subseteq \mathfrak{g}_k$ for all $k \geq 0$, nilpotent Lie algebras are solvable. The following result is easy to prove.

Proposition A.3 *Let $\mathfrak{g}$ be a Lie algebra.*

(i) *If $\mathfrak{g}$ is nilpotent, then so are all subalgebras and homomorphic images.*

(ii) *If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*

(iii) *If $\mathfrak{g}$ is nilpotent, then $Z(\mathfrak{g}) \neq 0$.* ■

As an example of nilpotent Lie algebras, we consider the subalgebra $\mathfrak{u}(n, \mathbb{C})$ of $\mathfrak{gl}(n, \mathbb{F})$ consisting of all strictly upper triangular matrices.

Given a $\mathfrak{g}$-module V with the corresponding representation ρ, we say that $x \in \mathfrak{g}$ is *nilpotent* on V if the linear map $\rho(x) : V \to V$ is nilpotent, i.e., $\rho(x)^n = 0$ for some integer $n > 0$. More generally, a subset S of $\mathfrak{g}$ is said to be *nilpotent* on V, if there exists a positive integer m such that $\rho(S)^m = 0$.

Theorem A.4 (Engel) *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$, and let $V \neq 0$ be a finite-dimensional $\mathfrak{g}$-module. If every element of $\mathfrak{g}$ is nilpotent on V, there exists $v \in V$, $v \neq 0$, such that $x \cdot v = 0$ for all $x \in \mathfrak{g}$.* ■

As an easy consequence of Theorem A.4, we have

Corollary A.5 *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V, \mathbb{F})$ be a representation of $\mathfrak{g}$, and assume each $\rho(x)$, $x \in \mathfrak{g}$, is nilpotent. Then there exists a full flag of subspaces V:*

$$V_1 \subset V_2 \subset \cdots \subset V_n = V, \ n = \dim V,$$

each properly contained in the next, such that the action of $\mathfrak{g}$ on each factor V_{i+1}/V_i, $1 \leq i \leq n$, is trivial. In particular, $\mathfrak{g}$ is nilpotent on V. ■

Theorem A.6 (Lie) *Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field $\mathbb{F}$, and let $V \neq 0$ be a finite-dimensional $\mathfrak{g}$-module. Then there exists $v \in V$, $v \neq 0$, such that v is a common eigenvector for all $x \in \mathfrak{g}$.* ■

We list some of the important consequences of Lie's theorem.

Corollary A.7 *Let $\mathfrak{g}$ be a solvable Lie algebra over a field $\mathbb{F}$, and let V be a finite-dimensional $\mathfrak{g}$-module. Then the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent on V.* ■

Corollary A.8 *Let $\mathfrak{g}$ be a solvable Lie algebra over a field $\mathbb{F}$. Then the derived ideal $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.* ■

Theorem A.9 (Cartan's Criterion of Solvability) *Given a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V, \mathbb{F})$, where V is a finite-dimensional linear space over a field $\mathbb{F}$, the following are equivalent.*

(i) $\mathfrak{g}$ *is solvable;*

(ii) $Tr_V(xy) = 0$ *for* $x \in \mathfrak{g}$ *and* $y \in [\mathfrak{g}, \mathfrak{g}]$. ■

Corollary A.10 *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$. *If* $\kappa_{\mathfrak{g}}(x, y) = 0$ *for all* $x, y \in \mathfrak{g}$, *then* $\mathfrak{g}$ *is solvable.* ■

A.3 Semisimple Lie Algebras

Recall (§A.2) that a nonzero Lie algebra $\mathfrak{g}$ is semisimple if $Rad(\mathfrak{g}) \neq 0$. Since the ideals in the derived series of $Rad(\mathfrak{g})$ are ideals of $\mathfrak{g}$, it follows easily that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}$ contains no nonzero abelian ideals.

For a linear subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$, define

$$\mathfrak{a}^{\perp} = \{x \in \mathfrak{g} : \kappa_{\mathfrak{g}}(x, \mathfrak{a}) = 0\}.$$

If $\mathfrak{a}$ is an ideal, then so is $\mathfrak{a}^{\perp}$ by the invariance of the Killing form $\kappa_{\mathfrak{g}}$ (see (A.1.1)).

Suppose now that $\mathfrak{a}$ is an ideal of a semisimple Lie algebra $\mathfrak{g}$. Then we have $\mathfrak{a} \cap \mathfrak{a}^{\perp} = (0)$ by Cartan's Criterion (Theorem A.9) applied to $\mathfrak{a}$, and this also shows that $\dim \mathfrak{g} = \dim \mathfrak{a} + \dim \mathfrak{a}^{\perp}$, from which we deduce $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$. Moreover, $Rad(\mathfrak{a}) = Rad(\mathfrak{g}) \cap \mathfrak{a} = (0)$ implies that $\mathfrak{a}$ is semisimple. Thus we have

Proposition A.11 *Let* $\mathfrak{a}$ *be an ideal of a semisimple Lie algebra* $\mathfrak{g}$. *Then* $\mathfrak{a}$ *is semisimple and* $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$. ■

If $\mathfrak{g}$ is semisimple, clearly $\kappa_{\mathfrak{g}}$ is nondegenerate by Proposition A.11. On the other hand, if $\mathfrak{a}$ is an abelian ideal of any $\mathfrak{g}$, and if $x \in \mathfrak{a}$, then for any $y \in \mathfrak{g}$, $ad(x) \circ ad(y)$ maps $\mathfrak{g}$ into $\mathfrak{a}$, and $\mathfrak{a}$ into 0, so that $\kappa_{\mathfrak{g}}(x, y) = Tr(ad(x) \circ ad(y)) = 0$, i.e., $\kappa_{\mathfrak{g}}(\mathfrak{a}, \mathfrak{g}) = 0$. Thus if the Killing form $\kappa_{\mathfrak{g}}$ is nondegenerate, then $\mathfrak{a} = (0)$, and since $\mathfrak{g}$ contains no nonzero abelian ideal, $\mathfrak{g}$ is semisimple. This proves

Theorem A.12 *A Lie algebra* $\mathfrak{g}$ *is semisimple if and only if its Killing form is nondegenerate.* ■

Theorem A.13 *If* $\mathfrak{g}$ *is semisimple, then* $ad(\mathfrak{g}) = Der(\mathfrak{g})$.

Proof. Let $\delta \in Der(\mathfrak{g})$. The map $x \mapsto Tr(\delta \circ ad(x)) : \mathfrak{g} \to \mathbb{F}$ is a linear function, and since $\kappa_\mathfrak{g}$ is nondegenerate (Theorem A.12), there exists an element $z \in \mathfrak{g}$ such that $\kappa_\mathfrak{g}(z, x) = Tr(\delta \circ ad(x))$ for all $x \in \mathfrak{g}$. We claim: $\delta = ad(z)$. Let $\eta = \delta - ad(x)$. Then, for all $x \in \mathfrak{g}$, we have

$$\begin{aligned} Tr(\eta \circ ad(x)) &= Tr(\delta \circ ad(x)) - Tr(ad(z) \circ ad(x)) \\ &= Tr(\delta \circ ad(x)) - \kappa_\mathfrak{g}(z, x) = 0. \end{aligned} \quad \text{(A.3.1)}$$

Now for $x, y \in \mathfrak{g}$

$$\begin{aligned} \kappa_\mathfrak{g}(\eta(x), y) &= Tr(ad(\eta(x)) \circ ad(y)) \\ &= Tr([\eta, ad(x)] \circ ad(y)) \\ &= Tr(\eta \circ ad(x) \circ ad(y)) - Tr(ad(x) \circ \eta \circ ad(y)) \\ &= Tr(\eta \circ ad(x) \circ ad(y)) - Tr(\eta \circ ad(y) \circ ad(x)) \\ &= Tr(\eta \circ ad([x, y])) = 0 \end{aligned}$$

by (A.3.1). Thus $\eta(x) = 0$ for all $x \in \mathfrak{g}$, i.e., $\delta = ad(z)$. ■

Using Proposition A.11, we can easily show

Theorem A.14 *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ contains simple ideals $\mathfrak{g}_1, \cdots, \mathfrak{g}_\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\mathfrak{m}$.* ■

Corollary A.15 *If $\mathfrak{g}$ is semisimple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.* ■

Let $\mathfrak{g}$ be a Lie algebra, and let V be a $\mathfrak{g}$-module and let ρ be the associated representation of $\mathfrak{g}$. The $\mathfrak{g}$-module V is called *simple* (or *irreducible*) if $V \neq 0$ and V has no $\mathfrak{g}$-modules other than (0) and V, and it is called *semisimple* (or *completely reducible*) if every $\mathfrak{g}$-submodule has a complementary $\mathfrak{g}$-submodule.

Theorem A.16 (H. Weyl) *If $\mathfrak{g}$ is a semisimple Lie algebra, then every finite-dimensional $\mathfrak{g}$-module is semisimple.* ■

Levi Decomposition

Theorem A.17 *(Levi) Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras, and assume that $\mathfrak{h}$ is semisimple. Then there is a morphism $\sigma : \mathfrak{h} \to \mathfrak{g}$ such that $\phi \circ \sigma = 1_\mathfrak{h}$.* ■

Given a Lie algebra $\mathfrak{g}$, suppose that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$. Then $\mathfrak{g}$ is isomorphic to the semidirect sum $\mathfrak{a} \oplus_\rho \mathfrak{b}$, where $\rho : \mathfrak{b} \to Der(\mathfrak{a})$ is given by $\rho(b)(a) = [b, a]$, $b \in \mathfrak{b}$ and $a \in \mathfrak{a}$. We shall simply say that $\mathfrak{g}$ is a semidirect sum of the ideal $\mathfrak{a}$ and the subalgebra $\mathfrak{b}$. Levi's Theorem above states that $\mathfrak{g}$ is a semidirect sum of $\ker(\phi)$ and a subalgebra isomorphic to $\mathfrak{h}$.

We apply the above discussion to the following situation. For a Lie algebra $\mathfrak{g}$, the quotient algebra $\mathfrak{g}/Rad(\mathfrak{g})$ is semisimple, and hence by Levi's theorem, $\mathfrak{g}$ is a semidirect sum of the radical $Rad(\mathfrak{g})$ and a semisimple subalgebra $\mathfrak{s}$. $\mathfrak{s}$ is called a *Levi factor* (or a *Levi subalgebra*) of $\mathfrak{g}$, and the decomposition $\mathfrak{g} = Rad(\mathfrak{g}) \oplus \mathfrak{s}$ is called a *Levi decomposition* of $\mathfrak{g}$. As we shall see in Theorem A.21 below, any two Levi factors are conjugate. For the conjugacy theorem we need the following proposition, which is also used several times in Chapter 4.

Proposition A.18 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field $\mathbb{F}$, and let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$. If V is any finite-dimensional $\mathfrak{g}$-module, then $[\mathfrak{g}, \mathfrak{r}]$ is nilpotent on V (or equivalently, if V is a semisimple $\mathfrak{g}$-module, then $[\mathfrak{g}, \mathfrak{r}]$ acts trivially on V).*

Proof. Let $\mathfrak{s}$ be a maximal semisimple Lie subalgebra of $\mathfrak{g}$ so that we have a semidirect sum $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$. Then

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{r} + \mathfrak{s}] = [\mathfrak{g}, \mathfrak{r}] + [\mathfrak{s}, \mathfrak{s}] = [\mathfrak{g}, \mathfrak{r}] + \mathfrak{s}$$

and it follows that

$$[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = ([\mathfrak{g}, \mathfrak{r}] + \mathfrak{s}) \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}].$$

For the second assertion, let V be any finite-dimensional $\mathfrak{g}$-module. We may obviously assume that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V, \mathbb{F})$. By Corollary A.7, $[\mathfrak{r}, \mathfrak{r}]$ is nilpotent on V. Suppose the assertion is false, and choose a maximal $\mathbb{F}$-linear subspace T of $[\mathfrak{g}, \mathfrak{r}]$ that contains $[\mathfrak{r}, \mathfrak{r}]$ and that is nilpotent on V. Then $T \neq [\mathfrak{g}, \mathfrak{r}]$, and pick an element $x \in [\mathfrak{g}, \mathfrak{r}]$ but $x \notin T$. We may assume $x = [y, z]$ where $y \in \mathfrak{g}$ and $z \in \mathfrak{r}$. The subalgebra $\mathbb{F}y + \mathfrak{r}$ of $\mathfrak{g}$ is solvable, and hence $[\mathbb{F}y + \mathfrak{r}, \mathbb{F}y + \mathfrak{r}]$ is nilpotent on V. In particular, $[y, z]$ nilpotent on V, and $[[y, z], T] \subset [\mathfrak{r}, \mathfrak{r}] \subset T$. By Corollary A.5, the subalgebra $\mathbb{F}[y, z] + T$ is nilpotent on V. But this contradicts the maximality of T. This shows that $T = [\mathfrak{g}, \mathfrak{r}]$, so that $[\mathfrak{g}, \mathfrak{r}]$ is nilpotent on V. ∎

Remark A.19 The maximum nilpotent ideal $\mathfrak{m}$ of $\mathfrak{g}$ consists of the elements $x \in \mathfrak{r}$ such that $ad_{\mathfrak{g}}(x)$ are nilpotent. The second assertion of Proposition A.18, in particular, implies that $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{m}$. It follows that the inner derivation $ad_{\mathfrak{g}}(x)$, for $x \in \mathfrak{g}$, maps $\mathfrak{r}$ into $\mathfrak{m}$. Below we shall show that *any* derivation of $\mathfrak{g}$ maps $\mathfrak{r}$ into $\mathfrak{m}$. ■

Proposition A.20 *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, and let $\mathfrak{r}$ (resp. $\mathfrak{m}$) be the radical (resp. the maximum nilpotent ideal) of $\mathfrak{g}$. Then every derivation of $\mathfrak{g}$ maps $\mathfrak{r}$ into $\mathfrak{m}$.*

Proof. Let δ be a derivation of $\mathfrak{g}$. We first show that there is a Lie algebra $\mathfrak{h}$ over $\mathbb{F}$ that contains $\mathfrak{g}$ as an ideal of codimension 1 such that $\mathfrak{h} = \mathfrak{g} \oplus \mathbb{F}x_0$ for some $x_0 \in \mathfrak{h}$ and $\delta(x) = [x_0, x]$ for all $x \in \mathfrak{g}$. We view the field $\mathbb{F}$ as a 1-dimensional Lie algebra, and let $\phi : \mathbb{F} \to Der(\mathfrak{g})$ be the morphism of Lie algebras defined by $\phi(a) = a\delta$ for $a \in \mathbb{F}$. Let $\mathfrak{h}$ be the semidirect sum of $\mathfrak{g}$ and $\mathbb{F}$ with respect to the action ϕ. Thus, for $(x, a), (y, b) \in \mathfrak{h}$, $[(x, a), (y, b)] = ([x, y] + a\delta(y) - b\delta(x), 0)$. We identify $\mathfrak{g}$ with a subgroup of $\mathfrak{h}$ by means of the canonical injection $x \mapsto (x, 0) : \mathfrak{g} \to \mathfrak{h}$, and let $x_0 = (0, 1) \in \mathfrak{h}$. Then $\mathfrak{g}$ is clearly an ideal of $\mathfrak{h}$ of codimension 1, and $[x_0, x] = [(0, 1), (x, 0)] = (\delta(x), 0) = \delta(x)$ for all $x \in \mathfrak{g}$. Let $\mathfrak{r}'$ denote the radical of $\mathfrak{h}$. By Proposition A.18 and Remark A.19, $[\mathfrak{h}, \mathfrak{h}] \cap \mathfrak{r}' = [\mathfrak{h}, \mathfrak{r}']$, and for all $y \in [\mathfrak{h}, \mathfrak{r}']$, $ad_{\mathfrak{h}}(y)$ is nilpotent. It follows that $ad_{\mathfrak{g}}(y)$ is nilpotent for all $y \in [\mathfrak{h}, \mathfrak{r}'] \cap \mathfrak{g}$. This shows that $[\mathfrak{h}, \mathfrak{r}'] \cap \mathfrak{g}$ is a nilpotent ideal of $\mathfrak{g}$, and $[\mathfrak{h}, \mathfrak{r}'] \cap \mathfrak{g} \subset \mathfrak{m}$. Now $\mathfrak{r} \subset \mathfrak{r}'$, and $\delta(\mathfrak{r}) = [x_0, \mathfrak{r}] \subset [\mathfrak{h}, \mathfrak{r}'] \cap \mathfrak{g} \subset \mathfrak{m}$. ■

By Proposition A.18, $ad(x)$ is nilpotent on $\mathfrak{g}$ for $x \in [\mathfrak{g}, Rad(\mathfrak{g})]$. Hence $\exp(ad(x)) \in Aut(\mathfrak{g})$, and is given by

$$\exp(ad(x)) = \sum_{i=0}^{m-1} (i!)^{-1} (ad(x))^i,$$

where $m = \dim \mathfrak{g}$. With this observation, we now state the conjugacy theorem.

Theorem A.21 (Malcev) *Let $\mathfrak{g}$ be a Lie algebra and let*

$$\mathfrak{g} = Rad(\mathfrak{g}) \oplus \mathfrak{s}$$

be a Levi decomposition of $\mathfrak{g}$. If $\mathfrak{t}$ is any semisimple subalgebra of $\mathfrak{g}$, then there exists an element $x \in [\mathfrak{g}, Rad(\mathfrak{g})]$ such that $\exp(ad_{\mathfrak{g}}(x))$ maps $\mathfrak{t}$ into $\mathfrak{s}$. In particular, any two Levi factors are conjugate. ■

Reductive Lie Algebras A Lie algebra over a field is called *reductive* if its adjoint representation is semisimple.

Theorem A.22 *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then the following are equivalent.*

(i) $\mathfrak{g}$ *is reductive;*

(ii) *The derived ideal* $[\mathfrak{g}, \mathfrak{g}]$ *is semisimple;*

(iii) $\mathfrak{g}$ *has a faithful semisimple representation;*

(iv) $Rad(\mathfrak{g})$ *coincides with the center of* $\mathfrak{g}$*.*

In this case, $\mathfrak{g}$ *is the direct sum of its center and* $[\mathfrak{g}, \mathfrak{g}]$*.*

A.4 Cartan Subalgebras

We continue to assume that the field $\mathbb{F}$ is of characteristic 0. Let V be a finite-dimensional linear space over an algebraically closed field $\mathbb{F}$. If $t \in End_{\mathbb{F}}(V)$, then V is the direct sum of all subspaces $V_a(t) = \ker(t - aI_V)^m$, where m is the multiplicity of a as a root of the characteristic polynomial of t. Each subspace $V_a(t)$ is t-stable, and $t|_{V_a(t)}$ is the sum of aI and a nilpotent endomorphism.

Let $\mathfrak{g}$ be a Lie algebra, and let $x \in \mathfrak{g}$. We now apply the above discussion to the map $ad\, x (= ad_{\mathfrak{g}}\, x) : \mathfrak{g} \to \mathfrak{g}$. Thus $\mathfrak{g}$ is a direct sum of the $ad\, x$-stable subspaces $\mathfrak{g}_a(ad\, x)$. For $a, b \in \mathbb{F}$, we have

$$[\mathfrak{g}_a(ad\, x), \mathfrak{g}_b(ad\, x)] \subset \mathfrak{g}_{a+b}(ad\, x) \tag{A.4.1}$$

and this, in particular, shows that $\mathfrak{g}_0(ad\, x)$ is a subalgebra of $\mathfrak{g}$.

We also have $\mathfrak{g} = \mathfrak{g}_0(ad\, x) \oplus \mathfrak{g}^x$, where $\mathfrak{g}^x$ denotes the sum of those $\mathfrak{g}_a(ad\, x)$ for which $a \neq 0$.

By a *Cartan subalgebra* of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, we mean a nilpotent subalgebra $\mathfrak{h}$ that coincides with its own normalizer. To show $\mathfrak{g}$ has a Cartan subalgebra, we call an element $x \in \mathfrak{g}$ *regular* if $\dim \mathfrak{g}_0(ad\, x) = min\{\dim \mathfrak{g}_0(ad\, y) : y \in \mathfrak{g}\}$. Then we have

Theorem A.23 *If x is a regular element of a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{F}$, then $\mathfrak{g}_0(adx)$ is a Cartan subalgebra of $\mathfrak{g}$, and all Cartan subalgebras are obtained in this way. In particular, $\mathfrak{g}$ has a Cartan subalgebra.* ■

Let $\mathfrak{h}$ be a Lie algebra over an algebraically closed field $\mathbb{F}$, and let $\mathfrak{h}^* = Hom_{\mathbb{F}}(\mathfrak{h}, \mathbb{F})$, the space dual to $\mathfrak{h}$. Let V be a finite-dimensional $\mathfrak{h}$-module with the associated representation ρ. For $\gamma \in \mathfrak{h}^*$, let V_γ denote the subspace of V consisting of all $v \in V$ such that, for all $x \in \mathfrak{h}$,

$$(\rho(x) - \gamma(x) I_V)^m(v) = 0,$$

for some integer $m > 0$. If $V_\gamma \neq 0$, then we say γ is a *weight* of the representation ρ, and V_γ is called the *weight space* corresponding to the weight γ. V is the direct sum of the weight spaces V_γ.

Now let $\mathfrak{h}$ be a Cartan subalgebra of a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{F}$ of characteristic 0. A weight γ of the representation of $\mathfrak{h}$ on $\mathfrak{g}$ which is the restriction to $\mathfrak{h}$ of the adjoint representation of $\mathfrak{g}$ is called a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and the corresponding weight space $\mathfrak{g}_\gamma$ is called the *root subspace.* We note $\mathfrak{h} = \mathfrak{g}_0$, and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \mathfrak{h}^*$.

Now for semisimple Lie algebras, we have

Theorem A.24 *Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbb{F}$, and let Δ denote the set of all nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then*

(i) $\mathfrak{h}$ *is abelian;*

(ii) $\dim \mathfrak{g}_\alpha = 1$ *for each* $\alpha \in \Delta$*;*

(iii) $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ *(direct sum).*

A.5 Campbell-Hausdorff Formula

$\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$ throughout this section. We introduce a local Lie group structure in a neighborhood of 0 in a (finite-dimensional) Lie algebra over $\mathbb{K}$ by means of the Campbell-Hausdorff series. For a detailed discussion, see, e.g., [2] (Chap. II) or [10] (Chapter X).

Campbell-Hausdorff Series The *Campbell-Hausdorff series* in the variables x, y is a formal power series $z(x, y)$ in two noncommuting variables x and y, given by

$$z(x, y) \quad = \quad \log(\exp x \cdot \exp y)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \Big(\sum_{\substack{p,q=0 \\ p+q>0}} \frac{x^p y^q}{p!q!} \Big)^k$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum \frac{x^{p_1} y^{q_1} \cdots x^{p_k} y^{q_k}}{p_1! q_1! \cdots p_k! q_k!},$$

where the summation in the second sum is taken over all possible collections $(p_1, \cdots, p_k, q_1, \cdots, q_k)$ of nonnegative integers subject to the conditions

$$p_{1+}q_1 > 0, \cdots, p_{k+}q_k > 0.$$

For any positive integer n, we let

$$z_n(x,y) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum \frac{x^{p_1} y^{q_1} \cdots x^{p_k} y^{q_k}}{p_1! q_1! \cdots p_k! q_k!},$$

where the integers p_i and q_i in the second summation satisfy the *additional* condition

$$p_1 + \cdots + p_k + q_1 + \cdots + q_k = n.$$

Then $z_n(x,y)$ is the homogeneous component of total degree n so that we have

$$z(x,y) = \sum_{n=1}^{\infty} z_n(x,y).$$

For example, we have

$$\begin{aligned} z_1(x,y) &= x + y \\ z_2(x,y) &= \frac{1}{2}xy - \frac{1}{2}yx \\ z_3(x,y) &= \frac{1}{12}x^2y + \frac{1}{12}yx^2 + \frac{1}{12}xy^2 + \frac{1}{12}y^2x - \frac{1}{6}xyx - \frac{1}{6}yxy. \end{aligned}$$

We now consider the $\mathbb{K}$-algebra $\mathbb{K}\{x,y\}$ of polynomials over $\mathbb{K}$ in the *noncommuting* variables x and y. The associative $\mathbb{K}$-algebra $\mathbb{K}\{x,y\}$ becomes a Lie algebra over $\mathbb{K}$ if we define $[\ ,\ ]$ by

$$[f,g] = f \cdot g - g \cdot f$$

for $f,\ g \in \mathbb{K}\{x,y\}$. Let $L\{x,y\}$ denote the Lie subalgebra of $\mathbb{K}\{x,y\}$ that is generated by x, y (i.e., the smallest Lie subalgebra of the Lie algebra $\mathbb{K}\{x, y\}$ that contains x and y). Note that every element of $L\{x, y\}$ is a polynomial obtained starting from x and y by the operations of addition, multiplication by the elements of $\mathbb{K}$, and the bracket operation $[\ \ ]$. It is known that each $z_n(x, y) \in L\{x, y\}$. For example, we have

$$\begin{aligned} z_1(x, y) &= x + y \\ z_2(x, y) &= \frac{1}{2}[x, y] \\ z_3(x, y) &= \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]]. \end{aligned}$$

In general, there is an explicit formula for $z_n(x, y)$ using the bracket $[\ \ ]$ in $L\{x, y\}$.

Local Lie Groups Defined on Lie Algebras Now let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{K}$. The $\mathbb{K}$-bilinear map

$$(X, Y) \mapsto [X, Y] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

is continuous with respect to a Euclidean norm, and there is a norm $|\ \ |$ on $\mathfrak{g}$ such that

$$|[X, Y]| \leq |X||Y|$$

for $X, Y \in \mathfrak{g}$. We fix such a norm in the sequel.

For $X, Y \in \mathfrak{g}$, let $z_n(X, Y)$ be the element of $\mathfrak{g}$ obtained from the homogeneous component $z_n(x, y)$ of the Campbell-Hausdorff series by substituting X and Y for x and y, respectively, and we then interpret the bracket operation in $\mathbb{K}\{x, y\}$ as the bracket operation in $\mathfrak{g}$. Then we have

Proposition A.25 *If $X, Y \in \mathfrak{g}$, the series $\sum_{n=1}^{\infty} z_n(X, Y)$ converges whenever $|X| + |Y| < \log 2$.* ■

This proposition shows that there exists a neighborhood of 0 in $\mathfrak{g}$ on which the series

$$\sum_{n=1}^{\infty} z_n(X, Y)$$

converges absolutely. The series $z(X, Y) = \sum_{n=1}^{\infty} z_n(X, Y)$ is called the *Campbell-Hausdorff* (C-H) *series* for the Lie algebra $\mathfrak{g}$. If we write

$X \circ Y$ for $z(X, Y)$, then the multiplication $(X, Y) \mapsto X \circ Y$ defines a local $\mathbb{K}$-analytic group on an open neighborhood of 0 in $\mathfrak{g}$, for which 0 is the identity element and $-X$ is the inverse element of X. This multiplication is called the *Campbell-Hausdorff multiplication.*

By a *local* $\mathbb{K}$*-analytic group* we mean a $\mathbb{K}$-analytic manifold X together with a distinguished element $1 \in X$, an open neighborhood U of 1 in X, and a pair of $\mathbb{K}$-analytic maps

$$\begin{aligned} (x, y) \mapsto xy : U \times U \to X; \\ x \mapsto x^{-1} : U \to U \end{aligned}$$

satisfying the following properties:

(i) There exists a neighborhood V_1 of 1 in U with $x1 = x = 1x$ for $x \in V_1$;

(ii) There exists a neighborhood V_2 of 1 in U with $xx^{-1} = 1 = x^{-1}x$ for $x \in V_2$;

(iii) There exists a neighborhood V_3 of 1 in U such that $x(yz)$, $(xy)z$ are both defined and $x(yz) = (xy)z$ for $x,\ y,\ z \in V_3$.

In this case, 1 is called the *identity element* of the local group and x^{-1} the *inverse* of x.

Suppose now that $\mathfrak{g}$ is a Lie algebra of an analytic group. The C-H series gives a direct relationship between the Lie algebra of a group and the local structure of that group. For the proof of the following theorem, see [10] (Theorem 3.1, p. 112).

Theorem A.26 *Let G be a real analytic group with Lie algebra $\mathfrak{g}$. Then there exists a neighborhood W of 0 in $\mathfrak{g}$ such that the map*

$$(X, Y) \mapsto X \circ Y : W \times W \to \mathfrak{g}$$

is defined, and satisfies

$$\exp(X \circ Y) = (\exp X)(\exp Y),$$

for $X,\ Y \in W$. ■

Appendix B

Pro-affine Algebraic Groups

Some basic definitions and properties of pro-affine algebraic groups are summarized here. For more details consult [14].

B.1 Hopf Algebras

Throughout this section, $\mathbb{F}$ will denote a field of characteristic 0.

We first recall that an $\mathbb{F}$-algebra may be defined as a triple (A, μ, u) consisting of an $\mathbb{F}$-linear space A, together with $\mathbb{F}$-linear maps

$$\mu : A \otimes A \to A \text{ and } u : \mathbb{F} \to A,$$

called the *multiplication* and the *unit* of A, respectively, such that the following two diagrams

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{I \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes I}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \tag{B.1.1}$$

and

$$\begin{array}{ccccc} \mathbb{F} \otimes A & \xrightarrow{u \otimes I} & A \otimes A & \xleftarrow{I \otimes u} & A \otimes \mathbb{F} \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & A & & \end{array} \tag{B.1.2}$$

are commutative.

Note that the first diagram indicates the associativity of μ, and the second diagram shows the unitary property of A, where the maps $\mathbb{F} \otimes A \to A$, and $A \otimes \mathbb{F} \to A$ are the natural isomorphisms. If (A, μ, u) and (A', μ', u') are $\mathbb{F}$-algebras, a morphism of $\mathbb{F}$-algebras $\phi : A \to A'$ is an $\mathbb{F}$-linear map such that the diagrams

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\phi \otimes \phi} & A' \otimes A' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ A & \xrightarrow[\phi]{} & A' \end{array} \tag{B.1.3}$$

and

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ & {\scriptstyle u}\nwarrow \quad \nearrow{\scriptstyle u'} & \\ & \mathbb{F} & \end{array} \tag{B.1.4}$$

are commutative. That an $\mathbb{F}$-algebra morphism is a map preserving the product as well as the identity is stated as the commutativity of the two diagrams above.

Formal dualization of this definition yields the notion of an $\mathbb{F}$-*coalgebra*. Thus an $\mathbb{F}$-coalgebra is a triple (C, γ, c) consisting of an $\mathbb{F}$-linear space C together with $\mathbb{F}$-linear maps $\gamma : C \to C \otimes C$ and $c : C \to \mathbb{F}$ called the *comultiplication* and the *counit* of C, respectively, such that the following diagrams are commutative.

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & C \otimes C \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma \otimes I} \\ C \otimes C & \xrightarrow[I \otimes \gamma]{} & C \otimes C \otimes C \end{array} \tag{B.1.5}$$

and

$$\begin{array}{ccccc} & & C & & \\ & \swarrow & \downarrow{\scriptstyle \gamma} & \searrow & \\ \mathbb{F} \otimes C & \xrightarrow[c \times I]{} & C \otimes C & \xleftarrow[I \otimes c]{} & C \otimes \mathbb{F} \end{array} \tag{B.1.6}$$

Given $\mathbb{F}$-coalgebras (C, γ, c) and (C', γ', c'), a morphism of coalgebras $\alpha : C \to C'$ is an $\mathbb{F}$-linear map such that the following diagrams

$$\begin{array}{ccc} C & \xrightarrow{\alpha} & C \\ \gamma \downarrow & & \downarrow \gamma' \\ C \otimes C & \xrightarrow[\alpha \otimes \alpha]{} & C' \otimes C' \end{array} \tag{B.1.7}$$

and

$$\begin{array}{ccc} & \mathbb{F} & \\ c \nearrow & & \nwarrow c' \\ C & \xrightarrow[\alpha]{} & C' \end{array} \tag{B.1.8}$$

are commutative.

For an $\mathbb{F}$-algebra (A, μ, u), and an $\mathbb{F}$-coalgebra (C, γ, c), the $\mathbb{F}$-linear space of all $\mathbb{F}$-linear maps $C \to A$ is denoted by $Hom_{\mathbb{F}}(A, C)$. For $\varphi,\ \psi \in Hom_{\mathbb{F}}(A, C)$, define $\varphi * \psi$ to be the composite:

$$C \xrightarrow{\gamma} C \otimes C \xrightarrow{\varphi \otimes \psi} A \otimes A \xrightarrow{\mu} A. \tag{B.1.9}$$

The map

$$(\varphi, \psi) \mapsto \varphi * \psi$$

defines an associative binary operation on $Hom_{\mathbb{F}}(A, C)$, called the *convolution*, and with this product $Hom_{\mathbb{F}}(A, C)$ becomes an $\mathbb{F}$-algebra for which the map $u \circ c$ is the identity element.

An $\mathbb{F}$-algebra (B, μ, u) which is also an $\mathbb{F}$-coalgebra (B, γ, c) is called an $\mathbb{F}$-*bialgebra* if γ and c are morphisms of $\mathbb{F}$-algebras (or equivalently, if μ and u are morphisms of $\mathbb{F}$-coalgebras).

Let $H = (H, \mu, u; \gamma, c)$ be an $\mathbb{F}$-bialgebra. With $A = H = C$, the convolution product of (B.1.9) is defined on $Hom_{\mathbb{F}}(H, H)$, with which $Hom_{\mathbb{F}}(H, H)$ becomes an $\mathbb{F}$-algebra. If the identity map $I_H : H \to H$ has an inverse with respect to the convolution product, the inverse of I_H is called the *antipode* of the bialgebra H. An $\mathbb{F}$-bialgebra H with an antipode is called a *Hopf algebra* over $\mathbb{F}$. If a bialgebra has an antipode, it is unique, and we denote it here by η. Remembering that $u \circ c$ is the identity element, the defining property of η is

$$\mu \circ (\eta \otimes I_H) \circ \gamma = u \circ c = \mu \circ (I_H \otimes \eta) \circ \gamma. \tag{B.1.10}$$

Group of $\mathbb{F}$-algebra Homomorphisms Let $H = (H, \mu, u; \gamma, c)$ be an $\mathbb{F}$-bialgebra, and view the field $\mathbb{F}$ as an $\mathbb{F}$-algebra in a natural way. Then the convolution of (B.1.9) is defined on $Hom_{\mathbb{F}}(H, \mathbb{F})$, and is given by

$$\varphi * \psi = m \circ (\varphi \otimes \psi) \circ \gamma,$$

for $\varphi, \psi \in Hom_{\mathbb{F}}(H, \mathbb{F})$, where m denotes the field multiplication $\mathbb{F} \otimes \mathbb{F} \to \mathbb{F}$, which is an $\mathbb{F}$-linear isomorphism. With this convolution, $Hom_{\mathbb{F}}(H, \mathbb{F})$ becomes an $\mathbb{F}$-algebra, in which the counit c becomes the identity element. Suppose now that H is a Hopf algebra with the antipode η. Then every $\mathbb{F}$-algebra morphism $\phi : H \to \mathbb{F}$ is invertible in the algebra $Hom_{\mathbb{F}}(H, \mathbb{F})$, with its inverse $\phi \circ \eta$. In fact, we have, using the commutativity of (B.1.3),

$$\begin{aligned} \phi * (\phi \circ \eta) &= m \circ (\phi \otimes (\phi \circ \eta)) \circ \gamma \\ &= m \circ (\phi \otimes \phi) \circ (I_H \otimes \eta) \circ \gamma \\ &= \phi \circ \mu \circ (I_H \otimes \eta) \circ \gamma \\ &= \phi \circ (u \circ c) \qquad \text{by } (B.1.10) \\ &= (\phi \circ u) \circ c = I_{\mathbb{F}} \circ c = c. \end{aligned}$$

Using the fact that the comultiplication γ is a morphism of $\mathbb{F}$-algebras, we can show directly that if $\varphi, \psi \in Hom_{\mathbb{F}-alg}(H, \mathbb{F})$, then $\varphi * \psi \in Hom_{\mathbb{F}-alg}(H, \mathbb{F})$. Thus for any Hopf algebra H, the set $Hom_{\mathbb{F}-alg}(H, \mathbb{F})$ consisting of all $\mathbb{F}$-algebra morphisms $H \to \mathbb{F}$ forms a subgroup of the group of units in the algebra $Hom_{\mathbb{F}}(H, \mathbb{F})$.

B.2 Pro-affine Algebraic Groups

From here on, we assume that the field $\mathbb{F}$ is algebraically closed and of characteristic 0, although some of the stated results may be true more generally.

Pro-affine Algebraic Varieties A *pro-affine algebraic variety* over $\mathbb{F}$ is a pair (V, A), where V is a set and A is an $\mathbb{F}$-algebra of $\mathbb{F}$-valued functions on V, satisfying the following conditions.

(i) A separates the points of V, i.e., given x, $y \in V$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$;

(ii) Every $\mathbb{F}$-algebra homomorphism $A \to \mathbb{F}$ is the evaluation map x° at an element $x \in V$:

$$f \mapsto f(x) : A \to \mathbb{F}.$$

The conditions mean that the canonical map

$$x \mapsto x^\circ : V \to Hom_{\mathbb{F}-alg}(A, \mathbb{F})$$

is the bijection.

If A is, in addition, finitely generated, (V, A) is the usual structure of an *affine algebraic variety* . Given a pro-affine algebraic variety (V, A), the elements of A are called *polynomial functions* on V, and A is called the *polynomial algebra* of V. Usually the algebra A of all polynomial functions on V, which is denoted by $P(V)$, is usually omitted.

Given two pro-affine algebraic varieties (U, A) and (V, B), a map $\alpha : U \to V$ is called a *polynomial map* if the map $f \mapsto f \circ \alpha$ (called the *comorphism* of α) takes B into A. The pro-affine algebraic varieties form a category in which morphisms are polynomial maps. This category has a finite product. To describe this, let (U, A) and (V, B) be affine algebraic varieties. Then the elements of $A \otimes B$ can be viewed as functions on $U \times V$ in such a way that we have

$$(f \otimes g)(u, v) = f(u)g(v),$$

$f \in A$, $g \in B$, $u \in U$, and $v \in V$.

A pro-affine algebraic variety V may be viewed as a topological space, the topology being the *Zariski topology* in which a closed set is the set of zeros of subsets of $P(V)$.

Pro-affine Algebraic Groups Suppose a group G is given with the structure of a pro-affine algebraic variety. If the multiplication

$$(x, y) \mapsto xy : G \times G \to G$$

and the inversion

$$x \mapsto x^{-1} : G \to G$$

are morphisms of pro-affine algebraic varieties, we call G a *pro-affine algebraic group* over $\mathbb{F}$.

Let G be a pro-affine algebraic group. The polynomial algebra $P(G)$ is a sub Hopf algebra of $R_{\mathbb{F}}(G)$, or equivalently $P(G)$ is a fully stable subalgebra of $R_{\mathbb{F}}(G)$ (cf. Proposition 2.10). In fact, if $x \in G$, the morphism $\rho_x : G \to G$, given by $\rho_x(y) = yx$, induces the comorphism $\rho_x^* : P(G) \to P(G)$, and hence $x \cdot f = \rho_x^*(f) \in P(G)$

for all $f \in P(G)$. This shows that $P(G)$ is left stable. Similarly, we show that $P(G)$ is right stable. We now show that $P(G) \subset R_{\mathbb{F}}(G)$. Let $f \in P(G)$ and let B be the subspace of $P(G)$ that is spanned by the translates $G \cdot f$. We need to show that B is finite-dimensional. We may assume that $f \neq 0$. The multiplication $m : G \times G \to G$ induces the comorphism

$$m^* : P(G) \to P(G) \otimes P(G).$$

Let

$$m^*(f) = \sum_{i=1}^{n} g_i \otimes h_i,$$

where $g_i, h_i \in P(G)$.

For any $x, y \in G$, we have

$$f(xy) = \sum_{i=1}^{n} g_i(x) h_i(y),$$

and hence

$$y \cdot f = \sum_{i=1}^{n} h_i(y) g_i$$

for $y \in G$. This relation shows that B is a subspace of the linear span of the g_i, $1 \leq i \leq n$, and hence B is finite-dimensional. Moreover, the comorphism $i^* : P(G) \to P(G)$ induced by the inversion $i : G \to G$ maps each $f \in P(G)$ to f', and hence we see that $P(G)$ is a fully stable subalgebra of $R_{\mathbb{F}}(G)$.

The discussion above enables us to define a *pro-affine algebraic group* over $\mathbb{F}$ as a pair (G, A), where G is a group and A is a sub Hopf algebra (i.e., fully stable subalgebra) of $R_{\mathbb{F}}(G)$, satisfying

(i) A separates the points of G;

(ii) every $\mathbb{F}$-algebra homomorphism $\phi : A \to \mathbb{F}$ is the evaluation at an element $x \in G$.

If (G, A) and (H, B) are pro-affine algebraic groups, a homomorphism $\phi : G \to H$ is called a *rational homomorphism* if it is a polynomial map. Pro-affine algebraic groups form a category in which morphisms are rational homomorphisms. If $\phi : G \to H$ is a morphism of pro-affine algebraic groups, then the comorphism $\phi^* : P(H) \to P(G)$ is a morphism of Hopf algebras.

A (Zariski) closed subgroup K of a pro-affine algebraic group G is called an *algebraic subgroup* of G. The restrictions of the elements of $P(G)$ to an algebraic subgroup K make up the algebra $P(K)$ of polynomial functions on the algebraic subgroup K, and $P(K)$ is isomorphic with $P(G)/\mathcal{I}(K)$, where $\mathcal{I}(K)$ denotes the annihilator of K in $P(G)$, i.e., the ideal of $P(G)$ consisting of all f such that $f(x) = 0$ for all $x \in K$. For any ideal J of $P(G)$, let $\mathcal{Z}(J)$ denote the annihilator of J in $G = Hom_{\mathbb{F}-alg}(P(G), \mathbb{F})$. If K is any subgroup of G, then $\mathcal{Z}(\mathcal{I}(K))$ is a Zariski closed subgroup of G that contains K as a dense subgroup. We call $\mathcal{Z}(\mathcal{I}(K))$ the *algebraic hull* of K. This is the smallest algebraic subgroup that contains K.

Proposition B.1 *Let G be a group, and for a fully stable subalgebra S of $R_{\mathbb{F}}(G)$, we define $\mathcal{G}(S) = Hom_{\mathbb{F}-alg}(S, \mathbb{F})$. Then $\mathcal{G}(S)$ is a group under the convolution* (see §B.1). *If every element $f \in S$ is viewed as an $\mathbb{F}$-valued function on $\mathcal{G}(S)$ by evaluation:*

$$f(x) = x(f), \ x \in \mathcal{G}(S),$$

then $(\mathcal{G}(S), S)$ becomes a pro-affine algebraic group. ■

If G is a pro-affine algebraic group, and if S is a fully stable subalgebra of $P(G)$, then the canonical map $\phi : G \to \mathcal{G}(S)$ that sends each $x \in G$ onto the evaluation of S at x is a morphism of pro-affine algebraic groups, and its image is Zariski dense in $\mathcal{G}(S)$.

Proposition B.2 *Let G be a pro-affine algebraic group over a field $\mathbb{F}$. For a left stable finite-dimensional subspace V of $P(G)$, we define $\phi : G \to GL(V, \mathbb{F})$ by $\phi(x)(f) = x \cdot f, \ f \in V$. Then ϕ is a morphism of pro-affine algebraic groups.*

Proof. For $g \in V$ and a $\mathbb{F}$-linear function λ on V, define

$$\lambda/g : GL(V, \mathbb{F}) \to \mathbb{F}$$

by

$$(\lambda/g)(\beta) = \lambda(\beta(g)), \ \beta \in GL(V, \mathbb{F}),$$

and for $x \in G$, let x' denote the linear function $g \mapsto g(x) : V \to \mathbb{F}$. Choose a basis $h_1, \cdots, h_m$ of V, and let $x_1, \cdots, x_m \in G$ such that $h_i(x_j) = \delta_{ij}$ (Lemma 2.7). Then $x'_1, \cdots, x'_m$ is a basis of V^* which is dual to the basis $h_1, \cdots, h_m$. The polynomial algebra $P(GL(V, \mathbb{F}))$ is generated by the functions λ/f, $\lambda \in V^*$, $f \in V$, together with $\frac{1}{D}$, where D denotes the determinant function $GL(V, \mathbb{F}) \to \mathbb{F}$. Since the map $(\lambda, f) \mapsto \lambda/f$ is $\mathbb{F}$-bilinear on $V \times V^*$, the elements x'_i/h_j together with $\frac{1}{D}$ generate $P(GL(V, \mathbb{F}))$. To show that ϕ is rational, it is therefore enough to show that the elements $(x'_i/h_j) \circ \phi$ and $\frac{1}{D} \circ \phi$ belong to $P(G)$. Indeed, for $x \in G$, we have

$$(x'_i/h_j) \circ \phi(x) = x'_i(\phi(x)(h_j)) = (x \cdot h_j)(x_i) = (h_j \cdot x_i)(x)$$

and

$$\frac{1}{D} \circ \phi(x) = \frac{1}{D \circ \phi(x)} = D(x^{-1}) = D \circ \iota(x),$$

where $\iota : G \to G$ is the inversion map. Since $h_j \cdot x_i,\ D \circ \iota \in P(G)$, we have $(x'_i/h_j) \circ \phi,\ \frac{1}{D} \circ \phi \in P(G)$. ∎

Pro-affine Groups as Projective Limits of Affine Groups
Let $\{X_i,\ i \in I\}$ be a family of topological spaces X_i, indexed by a directed set I, and assume that, for any pair $i, j \in I$ with $i < j$, there is a continuous map $\pi_i^j : X_j \to X_i$ such that π_i^i is the identity map on X_i for each $i \in I$, and that $\pi_i^k = \pi_i^j \circ \pi_j^k$ whenever $i, j, k \in I$ with $i < j < k$. Then the family (X_i, π_i^j) of the spaces X_i together with the maps π_i^j is called a *projective system* of topological spaces.

An affine algebraic group G may be given with a T_1-topology whose closed sets are the unions of finite families of cosets xH, where x ranges over G and H ranges over the family of algebraic subgroups of G. This is called the *coset topology* of G. With respect to the coset topology, G is compact and every morphism of affine algebraic groups is continuous, and is even closed (i.e., maps a closed set to a closed set).

We make use of this topology in the application of the following lemma (see [14]).

Lemma B.3 *Let (X_i, π_i^j) be a projective system of compact T_1 spaces. If each map $\pi_i^j : X_j \to X_i$ is closed, then the projective limit $X = \varprojlim X_i$ is nonempty. Moreover, if, for some fixed i, π_i^j is surjective for all $j \geq i$, then the natural projection $X \to X_i$ is surjective.* ∎

Let G be a pro-affine algebraic group, and, for two finitely generated fully stable $\mathbb{F}$-subalgebras S and T of $P(G)$ with $S \subset T$, the restriction map $h_S^T : \mathcal{G}(T) \to \mathcal{G}(S)$ is a surjective morphism of affine algebraic groups. Moreover, if R is a finitely generated fully stable subalgebra of $P(G)$ with $R \subset S \subset T$, then $h_R^T = h_R^S \circ h_S^T$ holds. Thus we have a projective system of affine algebraic groups $(\mathcal{G}(T),\ h_S^T)$, where T ranges over all finitely generated fully stable subalgebras of $P(G)$. Let $\varprojlim \mathcal{G}(T)$ denote the projective limit of the system $(\mathcal{G}(T),\ h_S^T)$. The restriction maps $G \to \mathcal{G}(T)$ for various T define a canonical isomorphism

$$G \cong \varprojlim \mathcal{G}(T). \tag{B.2.1}$$

In fact, the map is clearly a homomorphism, and it is an injection since $P(G)$ is the union of all finitely generated fully stable subalgebras T. To show that the map is surjective, let

$$z = (z_T)_T \in \varprojlim \mathcal{G}(T),$$

and define $\varphi : P(G) \to \mathbb{F}$ by $\varphi(f) = z_T(f)$, where T is a finitely generated fully stable subalgebra of $P(G)$ that contains f. If S is another finitely generated fully stable subalgebra that contains f, then

$$z_T(f) = z_{S \cap T}(f) = z_S(f).$$

Thus φ is well defined, and $\varphi \in \mathcal{G}(P(G)) = G$ corresponds to z. This proves that the canonical map $G \to \varprojlim \mathcal{G}(T)$ is surjective. By Lemma B.3, the canonical maps $G \to \mathcal{G}(T)$ for various T are all surjective.

Recall that if $\phi : G \to H$ is a morphism of affine algebraic groups, then $\phi(G)$ is an algebraic subgroup of H. We shall extend this to the pro-affine case. For this we first establish the following extension theorem.

Theorem B.4 *Let G be a pro-affine algebraic group. If S is any fully stable subalgebra of $P(G)$, the canonical map $\phi : G \to \mathcal{G}(S)$ is surjective.*

Proof. First assume G is an affine algebraic group, i.e., $P(G)$ is finitely generated. Since our assertion holds for the affine case, we may assume that S is *not* finitely generated. Let $\sigma \in \mathcal{G}(S)$. For any finitely generated fully stable subalgebra R of S, the restriction map $\phi_R : G \to \mathcal{G}(R)$ is surjective, and hence

$$Z_R = \{x \in G : \phi_R(x) = \sigma_R\}$$

is nonempty. Moreover, Z_R is an algebraic subset of G (in fact, it is the coset $x \ker(\phi_R)$, where $x \in Z_R$), and if R_1 and R_2 are finitely generated fully stable subalgebras of S, then $Z_{R_1 R_2} \subset Z_{R_1} \cap Z_{R_2}$. Thus the family of the Z_R has a finite intersection property, and hence $\bigcap_R Z_R$ is nonempty by the compactness of G in the Zariski topology. If $x \in \bigcap_R Z_R$, then $\phi(x) = \sigma$, since S is the union of the family of the finitely generated fully stable subalgebras R's, proving the assertion in this case.

Now we assume that G is an arbitrary pro-affine algebraic group, and we write $G = \varprojlim \mathcal{G}(T)$ as in (B.2.1). If every algebraic group $\mathcal{G}(T)$ is given the coset topology, the maps h_R^T are all closed. Let $\sigma \in \mathcal{G}(S)$, and, for any finitely generated fully stable subalgebra $T \subset P(G)$, let X_T denote the set of all $z \in \mathcal{G}(T)$ such that $\sigma_{T\cap S} = h_{T\cap S}^T(z)$. By the first part of our proof, the assertion is true for the affine case, and the canonical map $\mathcal{G}(T) \to \mathcal{G}(S \cap T)$ is therefore surjective. This, in particular, implies that X_T is nonempty, and it is a coset of the algebraic subgroup $\ker(h_{T\cap S}^T)$ in $\mathcal{G}(T)$. Moreover, if $T \subset R$, then $h_T^R(X_R) \subset X_T$. We may now apply Lemma B.3 to the projective system (X_T, h_T^R) to conclude that $\varprojlim X_T$ is nonempty. Since $\varprojlim X_T \subset \varprojlim \mathcal{G}(T) = G$, there is an element $x \in G$ that corresponds to σ, proving our assertion. ■

Now we are ready to present the following result which is well known for the affine groups.

Theorem B.5 *If $\phi : G \to H$ is a morphism of pro-affine algebraic groups, then $\phi(G)$ is an algebraic subgroup of H.*

Proof. $\mathcal{Z}(\mathcal{I}(\phi(G)))$ is the Zariski closure of $\phi(G)$, so that we must prove that $\mathcal{Z}(\mathcal{I}(\phi(G))) = \phi(G)$. Let $x \in \mathcal{Z}(\mathcal{I}(\phi(G))) \subset H$. Since $\mathcal{I}(\phi(G))$ is the kernel of the comorphism $\phi^* : P(H) \to P(G)$, and since x annihilates $\mathcal{I}(\phi(G))$, x induces an algebra homomorphism $\phi^*(P(H)) \to \mathbb{F}$. By Theorem B.4, this extends to an element $y \in G$, and we have $x = \phi(y)$, proving $x \in \phi(G)$. ■

Normal Subgroups and Quotient Groups Let G be a pro-affine algebraic group and let N be a normal algebraic subgroup of G. The N-fixed part $P(G)^N$ of $P(G)$ is a fully stable subalgebra of $P(G)$, and the group $\mathcal{G}(P(G)^N)$ may be identified with the quotient group G/N. Thus G/N is a pro-affine algebraic group with $P(G)^N$ as its polynomial algebra.

Let N and K be algebraic subgroups of a pro-affine algebraic group G with N normal in G, and assume that $G = K \cdot N$ is a semidirect product. We obtain a tensor product decomposition of $P(G)$ as follows. The restriction images $P(G)_K$ and $P(G)_N$ are the polynomial algebras of K and N, respectively. The inclusion $P(G)^N \subset P(G)$, composed with the restriction map $P(G) \to P(G)_K$, is an isomorphism of $\mathbb{F}$-algebras. Next, let $\pi : G \to N$ be the projection. If $f \in P(G)_N$, then $f \circ \pi$ is left K-fixed, and hence the map $f \mapsto f \circ \pi$ defines an isomorphism $P(G)_N \to P(G)^K$ with its inverse being the restriction morphism $P(G)^K \to P(G)_N$. Then the multiplication $(f, g) \mapsto fg : P(G)^N \times P(G)^K \to P(G)$ defines an isomorphism

$$P(K) \otimes P(N) \cong P(G)^N \cdot P(G)^K = P(G),$$

where $P(N) = P(G)_N$ and $P(K) = P(G)_K$ are canonically isomorphic with $P(G)^K$ and $P(G)^N$, respectively, as described above.

We say that a pro-affine algebraic group G is *connected (or irreducible)* if $P(G)$ is an integral domain. The opposite extreme is the notion of a pro-finite algebraic group. We say that a pro-affine algebraic group G is *pro-finite* provided that $P(G)$ coincides with the union of the family of finite-dimensional fully stable subalgebras, so that G is the projective limit of a system of finite groups. Every pro-affine algebraic group G has a normal connected algebraic subgroup G_0 of G such that G/G_0 is pro-finite.

Unipotent Subgroups Recall (§2.1) that, for an abstract group K, a representation $\rho : K \to GL(V, \mathbb{F})$ (and the corresponding G-module V) is called *unipotent* if the set of endomorphisms $\rho(x) - 1_V$, with $x \in K$, is nilpotent on V, or equivalently, if each $\rho(x) - 1_V$ is nilpotent (Theorem 2.4).

Let G be a pro-affine algebraic group over $\mathbb{F}$. A subgroup H of G is called *unipotent* if, for every finite-dimensional left G-stable subspace V of $P(G)$, the representation of H by left translations on V is unipotent. We have

Proposition B.6 *For a subgroup H of a pro-affine algebraic group G, the following are equivalent.*

(i) *H is unipotent;*

(ii) *The restriction to H of every rational representation of G is unipotent.*

Proof. (i)⇒ (ii): Given a rational representation $\rho : G \to GL(V, \mathbb{F})$ of G, $[\rho]$ is a bistable finite-dimensional subspace of $P(G)$. Since we assume that H is unipotent, the restriction to H of the representation of G by left translations on $[\rho]$ is unipotent. Now by Lemma 6.5 the left G-module V associated with ρ may be embedded as a submodule into the G-module $[\rho] \oplus \cdots \oplus [\rho]$ ($dim_{\mathbb{F}}(V)$-times). Since the action of H on $[\rho] \oplus \cdots \oplus [\rho]$ is unipotent, it follows that the action of H on V is also unipotent, proving (ii).

(ii)⇒ (i): Suppose that V be a finite-dimensional left H-stable subspace of $P(G)$. Then there is a finite-dimensional left G-stable subspace W of $P(G)$ that contains V. Then the restriction to H of the representation of G by left translations on W is unipotent by (ii), and hence the action of H on V is also unipotent, and H is unipotent. ■

There exists a normal unipotent subgroup of G, which is maximal in the sense that it contains *every* normal unipotent subgroup of G. This subgroup is denoted by G_u, and is called the *unipotent radical* of G. Since the closure of a unipotent subgroup of G is again unipotent, it follows that G_u is an algebraic subgroup of G.

Proposition B.7 *If $\phi : G \to H$ is a surjective morphism of connected pro-affine algebraic groups, then $\phi(G_u) = H_u$.* ■

Decomposition of Pro-affine Algebraic Groups Let G be a pro-affine algebraic group G. A subgroup H of G is called *reductive* if its representation by left translations on $P(G)$ is semisimple. From the definitions, it is clear that a subgroup H of G is reductive if and only if the restriction to H of every rational representation of G is semisimple. Just as for affine algebraic groups, we have the following decomposition theorem.

Theorem B.8 ([14], Th. 3.3) *Let G be a pro-affine algebraic group. There is a reductive algebraic subgroup K of G such that G is the semidirect product $G_u \cdot K$. Moreover, if L is any reductive subgroup of G, then there is an element $t \in [G, G_u]$ such that $tLt^{-1} \subset K$.* ■

Let G be a pro-affine algebraic group, and let $A = P(G)$. An element $f \in A$ is called a *semisimple element* if the representation of G by left translations on the G-stable subspace spanned by $G \cdot f$ is semisimple. Following the same argument as in §2.7, we see that the semisimple elements in A form a fully stable subalgebra A_s of A. Then we have $A_s = A^{G_u}$. In fact, if $f \in A_s$, then the restriction to the normal subgroup G_u of the action of G on V_f is semisimple. On the other hand, the action of G_u on V_f is unipotent by Proposition B.6. Thus the action of G_u on V_f is trivial, i.e., $f \in A^{G_u}$, proving that $A_s \subset A^{G_u}$. Now in order to show $A^{G_u} \subset A_s$, let K be a maximal reductive algebraic subgroup so that G has a semidirect decomposition $G = G_u \cdot K$. Let $f \in A^{G_u}$. Then the action of G_u on V_f is trivial, while the action of K on V_f is semisimple. Thus V_f is a semisimple G-module, i.e., $f \in A_s$, and $A^{G_u} \subset A_s$ is proved.

Proposition B.9 *If G is a pro-affine algebraic group, then G_u is exactly the fixer of $P(G)_s$ in G.*

Proof. Clearly G_u is contained in the fixer of $P(G)_s$ in G. Now we show that the fixer of $P(G)_s$ is contained in G_u. Let K be a maximal reductive algebraic subgroup of G so that $G = G_u \cdot K$ (semidirect product), and let $\pi : G \to K$ be the projection. Then π is rational, and hence $P(K) \circ \pi \subset P(G)$, or equivalently, $P(G) \circ \pi \subset P(G)$. Since G_u is normal in G, it is clear that $P(G) \circ \pi \subset P(G)^{G_u} = P(G)_s$. Now suppose $z \in G$ fixes all $f \in P(G)_s$. For all $f \in P(G)$, we have

$$f(\pi(z)) = z(f \circ \pi)(1) = (f \circ \pi)(1) = f(1),$$

and hence $\pi(z) = 1$, proving that $z \in G_u$. ■

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List of Notations